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HEDGING GOVERNMENT BONDS WITH FUTURES CONTRACTS:

A comparison of hedge ratio estimation methods

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Abstract
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PURPOSE OF THE STUDY

The objective of this thesis is to compare empirically the hedging performance of five different hedge ratio estimation methods that can be used with bond portfolios. The models are the regression-based method, duration, principal component analysis, risk-point method and combination hedge. The performance is measured using two key criteria: the remaining variance in the hedged portfolios and the cost of the hedge measured as the number of hedging transactions implied by the models. In particular, the hedging performance of duration is compared to methods that allow nonparallel changes in the yield curve. Futures contracts were used as the hedging instruments.

DATA

The data in this study comprises of yields and prices of the German government bonds that were originally issued as ten-year bonds and whose maturity was longer than one year at the end of the estimation period. In addition, data of fixed income futures prices, conversion factors and yield and price data for cheapest to deliver bonds were needed. The data set covers a time period that begins in January 1999 and ends in May 2002, including 176 weekly observations.

RESULTS

While single bonds were hedged using models that allow nonparallel yield curve shifts and using more than one hedging instrument, the remaining variance in portfolios could be decreased by 13%-27% compared to the duration hedge. In line with the initial hypothesis, the hedging costs also increased, by 4%-20%.

When a bond portfolio consisting of 15 bonds with differing maturities were hedged, the models that allow nonparallel yield curve shifts, and use more than one hedging instrument at a time, are able to decrease the remaining variance by 50-60% compared to the duration hedged portfolio. Surprisingly, the hedging costs were slightly lower than with duration.

KEYWORDS

Hedging, hedge ratio, futures contracts, duration

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1. Introduction

The return of a fixed income investment consists of two factors[1]: the yield of the bond, and the capital gains or losses caused by the interest rate fluctuations. Government bonds are often considered risk-free, but the concept 'risk-free' refers to the position being free from default risk. If a bond is sold before its maturity, the investment can yield negative returns due to the capital losses related to an increase in interest rates. In this sense, an investment in government bonds is far from being risk-free.

An increase in interest rates causes capital losses to a long position in government bonds. However, the value of the portfolio can be protected to a considerable extent by selling short an appropriate amount of relevant securities, for example government bonds or futures contracts on government bonds. A loss realized in one position, whether cash bond or futures position, will be offset by a profit on the other position.

The reason for hedging is that it can lead to an improved risk/return relationship. According to the modern portfolio theory, it is usually possible to construct many portfolios having the same expected return but with different variance of returns. If there are two portfolios with the same expected return, the one with the lower risk is clearly the better investment (Wilmott, 2001, pp. 186). Let's consider a bond investor, who would like to reduce the risk of his portfolio, in this case the risk of capital losses caused by interest rate fluctuations. One method used in practice is to simply decrease the average duration of the portfolio, which could be achieved by replacing long-maturity bonds with bonds that have shorter maturities, or alternatively by selling short futures contracts on the bonds. After the trades are executed, for example one third of the portfolio's bonds could be hedged and two thirds unhedged. In this situation the investor would prefer that the hedged bonds are truly hedged, i.e. the combined value of these long bonds and the corresponding short futures contracts would not fluctuate with the interest rates. However, because of the limitations of duration as a bond portfolio risk measure, the actual return on the hedged portfolio will likely deviate from the expected risk-free return. The best method to measure the risk involved in the bond position is

[1] Ilmanen (1996) divides an n-year zero-coupon bond's holding-period return over the next period in components as follows: h_n = [one-period forward rate] – [duration * expected yield change over the period] + [0.5 * convexity * (expected yield change)^2].

not trivial, which makes it difficult to choose the correct amount of securities to be sold short. This thesis concentrates on the question how to choose the optimal amount of futures contracts the investor should sell short in order to achieve the minimum variance for the hedged position. In addition, the aim is also to take into account the transaction costs incurred.

As mentioned, the most widely used technique to measure the risk and to calculate the correct amount of securities to be sold short lies on the concept of duration. However, when a long bond position is hedged using duration, it is implicitly assumed that the yield curve is flat, i.e. interest rates for all maturities are equal, and that all the changes in interest rates will be parallel, i.e. if interest rates change, rates for all maturities change by an equal amount. In addition, often a portfolio of bonds with differing maturities and cash flow structures should be hedged. The duration of the portfolio is the market value weighted average of the duration of the single bonds. If the portfolio is hedged using the average duration as the portfolio's risk measure, it is implicitly also assumed that the small yield change will occur at the same time in all the bonds in the portfolio, which seems unlikely to happen in practice.

The upside of using duration as the risk measure is its simplicity. In addition, it has proven to be a reasonable method to estimate the risk involved also in practice. However, according to Ilmanen (1992), there has been contradiction in academia over the effectiveness of the duration as a risk measure. E.g. Brennan and Schwartz (1983) and Ingersoll (1983) find that duration works as well as more sophisticated models. On the contrary, e.g. Gultekin and Rogalski (1984) and Elton, Gruber and Nabar (1988) find that multi-factor models outperform duration. But regardless of its performance, the assumptions that lay the foundation of the concept of duration are false, both theoretically and empirically. These problems are discussed next.

Grinblatt and Titman (2001) show why it is theoretically impossible that a flat yield curve and parallel yield shifts would exist for longer periods of time. These assumptions would allow an arbitrage opportunity, in which a low convexity portfolio could be sold short in order to finance a long position in a high-convexity portfolio. Although this combined position would be initially self-financed, it would create arbitrage profits when yields would change in any direction, due to the convexity differences in the portfolios.

In addition to the theoretical problems, the three following empirical findings do not comply with the assumptions of flat term structure and parallel yield shifts. First, it seems that the assumption of a flat term structure does not reflect the reality. Ilmanen (1996) states that the yield curve has been historically upward sloping about 90% of time. Second, yield curve shifts do not seem to be parallel. Jones (1991) analyzes yield curve shifts between 1979 and 1990 and finds that three types of yield curve shifts are not independent, with the two most common types of yield curve shifts being (1) a downward shift in the yield curve combined with a steepening of the yield curve and (2) an upward shift combined with a flattening of the curve. He also finds that 50 basis point (i.e. 0.50%) upward shift in the yield curve is consistent with 12.5 basis points flattening and 2 basis points less humbedness. He finds that these changes are typical for large yield changes. Third, it seems that large non-parallel shifts of the yield curve are related to large movements of the level of interest rates. Fung and Hsieh (1996) study particularly the effect of larger yield curve movements and their effects to risk management of fixed income portfolios. The authors find that extreme changes in yield spreads are correlated with extreme changes in the short rate and the 3 - 6 month yield spread. Therefore large movements in yield curve shape are related to large movements of the level of interest rates. This fact raises some important risk management issues for market neutral strategies, since the fact that level and spread events were correlated would mean that zero duration portfolios have directional exposure during extreme moves in interest rates.

Clearly there seems to be problems in duration as the portfolio risk measure. Many academics and practitioners have considered other methods to measure the yield curve risk in the bond portfolios. They have come up with several more advanced methods to take into account also yield curve shape changes, i.e. so called nonparallel shifts. A nonparallel shift of interest rates means that yields for all maturities do not change by the same amount. In practice, this concept usually means either of the following: (1) a slope change, i.e. the yield spread change between a short and a long maturity bond or (2) a curvature change, i.e. increase or decrease of the humpedness of the yield curve in the intermediate sector. The studies include Litterman and Scheinkman (1991), Ho (1992), Fabozzi and Dattatreya (1995), Chambers, Carleton and McEnally (1988) and Leschhorn (2001).

Litterman and Scheinkman (1991) study yield curve dynamics using a statistical method called principal component analysis (PCA). They show that the three first principal components are able to explain almost all the variance in the yield curve. In addition, these components can be intuitively interpreted as the level, steepness and curvature movements of the yield curve. Barber and Copper (1996) apply PCA to historical spot rate changes and show how a portfolio can be immunized in a traditional sense using the method. In this context, hedging and risk management become a matter of managing the portfolio's exposure to the principal components. In their model, the first three principal components are able to explain 97% of the variance.

Dattatreya and Fabozzi (1995) present a risk-point method for measuring and controlling yield curve risk. The innovation in the model is that the interest rate sensitivity of the bond to be hedged is determined in relation to the hedging instruments. The authors claim that the method is useful especially in determining and hedging against the yield curve risk. However, it seems that the model's hedging performance is still empirically untested.

In a recent study, Leschhorn (2001) presents a method called combination hedge to generalize the traditional duration model to take into account also nonparallel changes in the term structure of interest rates. This model is computationally simple but the author claims it is also able to take into account changes in the yield curve slope and curvature. In addition, the model does not lie on the basis of historical data or covariance matrices, which makes the calculations more easily understandable. An advantage of a simple yield curve model is that the correctness of the model's assumptions can be easily assessed, whereas it is often more difficult to decide whether a highly sophisticated model based on historical data will yield reasonable results.

The portfolio could be hedged by selling short bonds and investing the proceeds risk-free, or alternatively by using derivatives. The futures market on government bonds is very liquid for e.g. US or German government bonds, and the transaction costs are low, so using futures contracts seems to be a good alternative. However, hedging using futures contracts also causes additional exposures to the combined position, and these factors have to be taken into account and understood in order to hedge the position properly. Plona (1997, pp.189) provides the following reasons why to use futures contracts instead of bonds as a hedging instrument: "Futures are perhaps most useful in the context of a portfolio of bonds. Portfolios will be shown here to have parallel risk and return measurements as bonds. Futures, in a portfolio context, are a tool of separating the question of interest rate risk from the question of securities selection. Securities can be selected based on long-term funding requirements, current income potential, duration, convexity or liquidity considerations. A futures position overlaid upon the portfolio will alter its exposure to the changes in the interest rate environment while leaving the underlying cash flow structures intact."

In this study, the long bond portfolios are hedged using futures contracts as hedge instruments.

1.1. The motivation and objectives of the study

What does this study add to the existing literature? It seems obvious that the assumptions behind duration are false. However, the method is widely used. The study finds out how large error is made when the hedging decision is based on duration, and whether it could be possible to improve the hedge by using different models to estimate the hedge ratio.

In addition, although several models have been presented to take also the effect of nonparallel yield curve shifts into account, there is so far little empirical evidence that would support or be against the use of these models, especially in a context of using futures contracts as the hedge instruments. I will compare the hedging performance of these more advanced models to the performance of the traditional duration and regression-based hedges. Furthermore, there might be significant performance differences between the models that are able to take nonparallel shifts into account. Out of the alternatives available, I will choose the models that seem to be most effectively adaptable while hedging with futures contracts.

So far, the existing empirical research on the hedging methods is extensively concentrated on the US data. Especially due to the advent of euro, there could be interest in a study with a European data set. The investors can invest in a very large pool of government bonds without incurring the risk of foreign exchange fluctuations. The correlations between the yield changes of the government bonds of the different euro zone countries can be expected to be quite high, also in the future. Since there exists a highly liquid market for futures contracts on two-year, five-year and ten-year German government bonds, these contracts could probably be successfully used to hedge also portfolios of government bonds of other euro zone countries. This phenomenon increases the interest in the study of the hedging performance of these contracts.

The empirical study is performed as follows: I will simulate a situation, in which an investor has a long government bond portfolio that is hedged by selling short futures contracts on government bonds. The instruments chosen for the study are German government bonds and the Eurex-traded two-year, five-year and ten-year futures contracts on these bonds. I will try to achieve combined portfolios of long bonds and short futures contracts that would have a zero variance and therefore would yield the risk-free return. Between 1/1/1999 and 5/31/2002[2] I will estimate hedge ratios weekly using five different hedge ratio estimation methods, change the amount of short futures contracts in the portfolios accordingly, and record the gains and losses observed in the bond and futures portfolios. The objective is to compare the differences in the hedging performance of the models, and also compare the amount of futures trades that the models would have implied, in order to be able to compare the costs of these hedges.

1.2. Structure of the study

The structure of this paper is as follows. In section 2, I will present the theory behind the different yield curve risk estimation methods used in the study and present also the existing empirical findings that back up their use. Section 3 describes the properties of fixed income futures contracts and the implications of hedging with them. In section 4 the data and the methodology used in the study are presented as well as the hypothesis for the study. Section 5 describes the empirical findings. Finally, conclusions are stated in Section 6.

2. Hedging and hedge ratio estimation methods

[2] The notation of dates in this study is of the form mm/dd/yy.

The traditional approach to hedging with futures contracts as originally presented by Johnson (1960) for commodities, or Ederington (1979) for financial futures is based on risk reduction alone and ignores investor's expected return from holding the portfolio.

However, only a totally risk averse investor can make an optimal hedging decision without taking the impact on both risk and return into account. As Cecchetti, Cumby and Figlewski (1988) point out, real world hedgers are very aware of this tradeoff. They may hedge partially or selectively, and remain exposed to market risk on part of their position or part of time. In many cases, potential hedgers decide that hedging is not attractive for them because it is "too expensive".

Cecchetti, Cumby and Figlewski (1988) present an approach to hedging with futures contracts, which takes into consideration also expected return and time variation in the distribution of cash and futures price changes. Traditional hedging methods assume that the objective is to minimize risk, not to maximize expected utility, which also depends on expected returns. In addition, according to the authors, the joint distribution of cash and futures price changes and therefore the hedge ratio is estimated incorrectly since there is no adjustment for the fact that it varies substantially over time. Due to the time variation in the joint distribution of returns a regression employing past data will not correctly estimate the current risk minimizing hedge ratio.

The problem of the expected utility maximization is that an assumption of the utility function of the investor is needed, and due to practicalities log-function is usually chosen, which may not reflect the reality either. In this study, the approach is the traditional: the objective is to minimize the variance of the hedged portfolio.

2.1. Regression as a hedging tool

Johnson (1960) introduced minimum variance hedge ratios. Figlewski (1985) presents the hedge ratio estimation with regression as follows. The long bond position is being hedged using h units of futures contracts. Expected profit and variance of the combined position are

$$E(r) = (P_1 - P_0) - h \cdot (F_1 - F_0) \tag{1}$$

$$Var(r) = \sigma_p^2 + h^2 \cdot \sigma_f^2 - 2 \cdot h \cdot \rho \cdot \sigma_p \cdot \sigma_f \tag{2}$$

where P_1 is bond price at time 1

P_0 is bond price at time 0

F_1 is futures price at time 1

F_0 is futures price at time 0

h is the number of futures sold

σ_p, σ_f are the standard deviations of the returns of the bond and the futures contract

ρ is the correlation coefficient between the bond and futures returns

The risk minimizing h can be determined by taking a first derivative of the function of variance in respect to h. The risk minimizing h is then

$$h^* = \frac{\operatorname{cov}(P_1 - P_0, F_1 - F_0)}{\operatorname{var}(F_1 - F_0)} \tag{3}$$

In practice, the h can be estimated using regression. The dependent variable is the change in the price of bond to be hedged and the independent variable the change in the futures price. The estimated beta coefficient is by construction of regression

$$\beta = \frac{\operatorname{cov}(y, x)}{\operatorname{var}(x)} \tag{4}$$

which is the same as the h in risk minimizing hedge, when y represents changes in price of the bond to be hedged and x the changes in price of futures contract. Therefore, beta coefficient of a regression can be used as the hedge ratio. Then the estimated beta can be thought as a portfolio's beta coefficient relative to the futures contract.

Figlewski (1983) notes that the variance of futures returns is influenced by two random variables: total return on the underlying portfolio and the change in the basis[3] between the futures contract and the underlying security. These will naturally affect the risk minimizing hedge ratio as well. In the special case when the hedge is to be held until the futures contracts expire, so that the change in the basis is also nonstochastic, the effect of these terms disappears, leaving the optimal hedge ratio equal to the covariance of the portfolio to be hedged and the underlying security of the futures contract divided by the variance of the underlying security. The risk minimizing hedge ratio in this special case is portfolio's beta in respect to the underlying security of the futures contract. When basis is volatile over short periods, using beta as the hedge ratio is unlikely to be optimal, unless the position is to be held until maturity of the futures contracts.

An influential early work on reducing interest rate risk with financial futures is Ederington (1979). He uses similar regressions than Johnson (1960) had used with commodities data. Ederington (1979) examines the hedging performance of the futures markets in financial securities by using a basic portfolio model. A conclusion he reaches is that two week hedges using 90-day Treasury bill futures are rather ineffective in reducing exposure to price change risk. However, in his response to Ederington (1979), Franckle (1980) comments that the weaker than expected hedging performance was likely caused by problems in the data that was used in Ederington (1979) study. Ederington used weekly average prices for the T-Bill contracts although his futures price data were from Fridays each week.

Ederington (1979) recognizes that the value of the beta should be adjusted because of the decreasing maturity of the T-Bill during the period of the hedge, but he makes no estimates of the necessary adjustments. In his empirical study he assumes that he has a 90-day T-Bill in

[3] Basis refers to the difference between the cash and futures prices. The concept is discussed in detail in the next section.

the beginning and in the end of the hedge, although the maturity of the T-bill in the end of the hedge is really 76 days. When Franckle (1980) makes adjustments to the hedge ratio to take this fact into account, the reduction in the variance of the hedged position is approximately the same as without the correction.

Koutmos and Pericli (2000) study hedging mortgage bonds with two-year, five-year, ten-year and thirty-year US Treasury futures contracts using multiple regression. Their out-of-the-sample results suggest unexpectedly that hedging with a single instrument would yield a better hedge than using the same contract together with another futures contract with different maturity. The result seems counterintuitive and the authors locate the source of the problem as the high correlation between returns of different maturity futures contracts. The correlation coefficients of the explanatory variables vary between 0.84 and 0.93, which leads to serious multicollinearity and therefore inefficient estimates of betas, which are the hedge ratios.

Due to the experiences in the study by Koutmos and Pericli (2000), I will not use multiple regression in the hedge ratio estimation. The bond or portfolio is hedged using one hedging instrument at a time, and this futures contract is the one with the duration closest to the duration of the bond or portfolio that is hedged.

2.2. Duration

The traditional and by far the most widely used method for controlling interest rate risk is the concept of duration, which is originally developed by Macauley in 1938 and Hicks in 1939, but they were apparently unaware of each other and derived the concept in different ways. Macauley (1938) states that "for a study of the relations between long and short term interest rates, it would seem highly desirable to have some adequate measure of 'longness'. Let's use the word 'duration'." Bierwag and Kaufman (1978) emphasize that it is important to note that the measure was developed solely on intuitive grounds to obtain a better single valued measure of the life of a payments stream. Hicks (1939) derives the method as an elasticity of a capital value of a payments stream with respect to interest rate.

The traditional theory of immunization as formalized by Fisher and Weil (1971) defines the conditions under which the value of an investment in a bond portfolio is protected against changes in the level of interest rates. The specific assumptions of this theory are that the portfolio is valued at a fixed horizon date, that there are no cash inflows or outflows within the horizon, and that interest rates change only by a parallel shift in the forward rates. Under these assumptions, a portfolio is said to be immunized if its value at the end of the horizon does not fall below the target value, where the target value is defined as the portfolio value at the horizon date under the scenario of no change in the forward rates. The main result of this theory is that immunization is achieved if the duration of the portfolio is equal to the length of the horizon. (Fong and Vasicek, 1984).

The assumption that interest rates can only change by a parallel shift has been a subject of considerable concern. For example Bierwag (1977) and Bierwag and Kaufman (1977) have postulated alternative models of interest rate behavior. A limitation of this approach is that the portfolio is protected only against the particular type of interest rate change assumed. Fong and Vasicek (1984) take another approach. They note that if it would be found out that a portfolio exposure to an arbitrary type of interest rate change were determined by some characteristic of the portfolio, then this characteristic could be considered a measure of immunization risk. By minimizing this risk measure, the portfolio could be structured to have as little vulnerability as possible to any interest rate movement. They call the characteristic they find as M^2. It is a weighted variance of time to payments around the horizon date. By minimizing the M^2, the immunized portfolio has a minimum variance regardless of the type of the interest rate shift. In this study, the hedging method based on principal component analysis could be categorized in this latter category, in which some characteristics of the portfolio are estimated and the hedging consists in practice of minimizing these factors in the hedged portfolios.

Ilmanen (1992) states that despite the widespread use of duration, academic research yields conflicting evidence on its quality. Gultekin and Rogalski (1984) find that duration explains about 50% of the cross-sectional return variation among government bonds in their 1947-1976 sample. Elton, Gruber and Nabar (1988) find that duration explains 62% of the variation, using government bond portfolios. However, Litterman and Scheinkman (1991) are able to explain almost 90% of the variation with their first component that closely resembles duration in their principal component analysis model. In his study, Ilmanen (1992) finds that duration's explanatory power has increased over time and that during the 1980's duration has explained 80% to 90% of the return variance for government bonds. Ilmanen (1992) provides two reasons for the improvement, first, overall yield volatility has increased, which strengthens the systematic component in bond returns relative to the unsystematic component, and second, parallel yield shifts have become more important relative to changes in the shape of the term structure.

The Macauley duration is defined as follows:

$$D = \frac{1}{P} \cdot \left[\sum_{j=1}^{T} \frac{C_j \cdot j}{(1+y)^j} \right] \tag{5}$$

where D is Macauley duration, P is price of the bond, C_j is the annual coupon payment, y is yield to maturity, and j is the time to the coupon payment. The interpretation of Macauley duration is easy: the higher the duration, the higher the interest rate risk. However, its usefulness was greatly improved, when the concept of modified duration was introduced. From Macauley duration, the modified duration can be calculated as follows:

$$MD = \frac{D}{1+y} \tag{6}$$

where MD is modified duration, D is Macauley duration, and y is yield. Modified duration links a bond's Macauley duration and its actual price volatility together for small changes in interest rates:

$$\frac{\Delta P}{P} = -MD \cdot \Delta y \tag{7}$$

where $\frac{\Delta P}{P}$ is a percentage change in a bond price, *MD* is modified duration and Δy is a change in a bond's yield. However, this is only an approximation. The price-yield relationship of a bond is convex, not linear, so the former gives a good approximation of a bond price change only to small changes in interest rates. Duration of a portfolio needs frequent rebalancing since it shortens with the maturity of a bond but not with the same pace. The duration shortens more slowly than the passage of time. In general, three characteristics of a bond affect to the duration. These are the term to maturity, the coupon, and the yield of the bond. Duration increases with maturity, decreases with increasing coupons and increases when yields go down.

Sundaresan (1997) denotes the modified duration as follows

$$MD = \frac{\partial P}{\partial y} \cdot \frac{1}{P} \tag{8}$$

In this notation it is easier to see that the duration is a first derivative of the bond price in respect to the yield. In addition to the bond price also the duration changes with yield, which can be taken into account by using the second derivative of the price in respect to the yield, the convexity, which is denoted as

$$Cx = \frac{1}{2} \cdot \frac{\partial^2 P}{\partial y^2} \cdot \frac{1}{P} \tag{9}$$

The following equation describes the price change of a bond, when both the duration and the convexity are taken into account in the calculation of the price change caused by a yield shift:

$$\frac{\Delta P}{P} = -MD \cdot \Delta y + \frac{1}{2} \cdot Cx \cdot (\Delta y)^2 \tag{10}$$

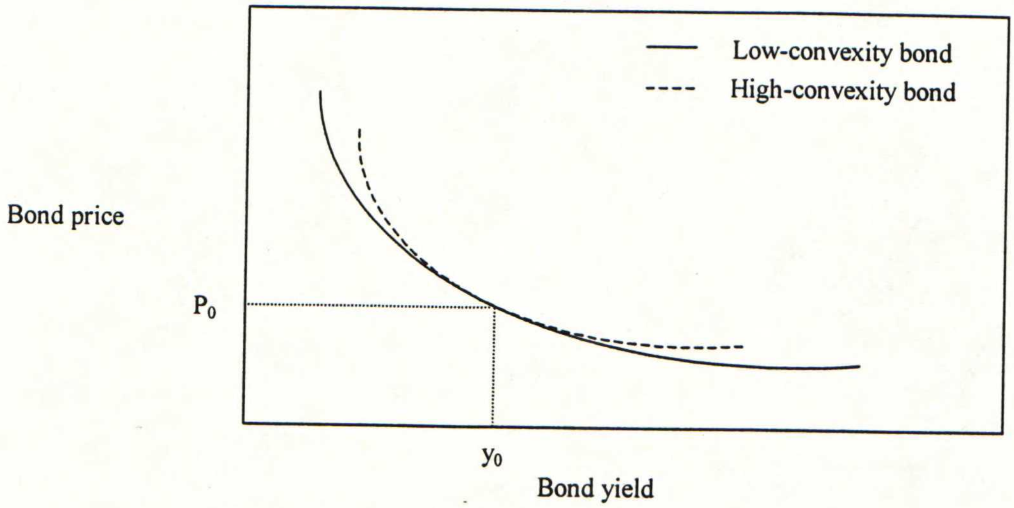
Holding maturity and yield constant, the convexity decreases as the coupon increases. Convexity increases with duration. (Sundaresan, 1997, pp. 149)

The duration of a bond portfolio can be defined as a weighted average of the durations ~~of~~ the individual bonds in the portfolio, with the weights being proportional to the bond prices. Equation 10 then approximates the price change of the portfolio in respect to a small yield change in all the bonds in the portfolio. It is important to realize that when duration is used for bond portfolios, there is an implicit assumption that the yields of all bonds will change by the same amount. When the bonds have differing maturities, this happens only when there is a parallel shift in the zero-coupon yield curve. Therefore the equation 10 provides an estimate of the impact on the price of a bond portfolio of a parallel yield shift in the zero curve. (Hull, 2002, pp.121)

Grinblatt and Titman (2001, pp. 842) provide an example of why the term structure cannot be flat in practice. They note that if the term structure would be flat, it would always be possible to create two portfolios with the same market value and a dollar value of a one basis point decrease in yield, but with differing convexities. These portfolios have the same yield to maturity, since all the yields are equal. If it would be possible to construct two portfolios that have characteristics like the ones described in figure 1, there would be an arbitrage opportunity. By going long in the high-convexity portfolio and selling short the low-convexity portfolio, one achieves an investment combination that is self-financing and – for any size move in the interest rate – has a positive value immediately after such move. Therefore relationships depicted in figure 1 are unlikely to exist in reality.

Figure 1: Convexity differences between bonds

The figure presents an example of two bond portfolios that have the same yield and price, but different convexities.



A short summary of the factors causing ineffectiveness to a duration hedge:

- In practice the term structure is not flat, and when it moves, the moves are not parallel, therefore the underlying assumptions behind bond portfolio immunization do not hold
- Duration matching does not immunize a portfolio against nonparallel shifts in the zero yield curve.
- The duration relationship applies only to small changes in yields. Duration is the first derivative of the convex price/yield relationship. For larger yield changes also the second derivative, namely convexity, should be taken into account

2.3. Principal components analysis

"The conventional approach of using duration to assess risk implicitly assumes perfect correlation between all points on the yield curve with no term structure of volatility. In contrast, the key rate duration approach works as if there is no correlation across the curve. Principle components analysis (PCA) tries to bridge that gap by taking account of the correlation and volatility structure of the yield curve. Market participants think of yield curve movements in terms of three components: level shift, slope change and a curvature change. With PCA this view point can be formalized." (Baygün, Showers, Cherpelis, 2001)

Principal components analysis (PCA) is a purely statistical technique that is primarily used to reduce the effective dimensionality of a problem. In other words, if some of the original variables, in this study the interest rate changes of different maturities, are highly correlated, they are 'effectively saying the same thing'. In this case the variation in the data can possibly be described by using just a few variables, called principal components. The usual objective is to see if the first few components account for most of the variation in the original data (Chatfield, Collins, 1980).

The method has also gathered some criticism when applied to financial data. It is computationally expensive and slow and the optimization procedure may intensify the effect of inexact assumptions so the calculations may lead to erratic results (Derman 1997). In addition, portfolio managers don't like to rely on calculations based on historical data because there is no guarantee the market will behave in the future as it has in the past and, in addition, the results can be quite dependent on the choice of the data in the sample (Leschhorn, 2001).

PCA was first introduced as a bond portfolio management tool by Litterman and Scheinkman (1991). They show that the three first principal components can explain almost all the variance in the yield curve. In addition, these components can be intuitively interpreted as the level, steepness and curvature movements of the yield curve. Barber and Copper (1996) use PCA to determine the best set of fundamental directions in which to anticipate spot rate changes and show how that information can be used to immunize a liability stream. In this context, hedging and risk management become a matter of managing the portfolio's exposure to the principal components. In their model, the first three principal components are able to explain 97% of the variance. Litterman and Scheinkman (1991) provide some empirical evidence on their model's hedging ability. They state that a single bond hedged with a portfolio of other bonds using PCA to calculate the hedge ratios results in a 28% average reduction in the variance on the hedged portfolio compared to duration hedge. Barber and Copper (1996) do not provide empirical evidence on the hedging performance of their model.

In the determination of the hedge ratios I use the methods presented by Barber and Copper (1996). Barber and Copper rely on Reitano (1991, 1992), who has developed a method for classifying term structure shifts. A given shift is thought of as a vector whose elements correspond to rate changes at different maturity dates. Viewed in this manner, Fisher-Weil immunization protects a bond portfolio against a shift in the direction of a vector with all components equal. Different duration measures given in the literature are designed to protect portfolios against term structure shifts in particular direction. Given that we do not know a priori the direction in which the term structure will shift, we are faced with a practical problem of choosing the best single direction in which to anticipate a shift. A more general model allows term structure to shift in multiple directions. Then an immunized portfolio must be protected against shifts in each of the fundamental directions. The problem is determining the set of fundamental directions. The only guide we have is the history of term structure movements and the principal components analysis is used to find out the best set of fundamental directions.

The Barber and Copper (1996) method differs from Litterman and Scheinkman (1991) in that the latter study uses data on excess returns on a risk-free rate and describe one-period rates of return on synthesized zero coupon bonds. Using the PCA model they construct riskless portfolios with zero factor loadings i.e. no exposure to the principal components. In this article zero yield changes are used instead of excess returns.

2.3.1. Single direction models

Interest rate changes of different maturities are positively correlated. At first the assumption is that the changes are perfectly correlated. It is supposed that

$u(s)$	is known function of maturity date
h	is a random variable
$P_i(h)$	is the price of €1 promised at date t_i
	$= \exp[-r(t_i)\cdot t_i]$
$r(t_i)$	is spot rate of interest
C_i	is cash flow stream

Value of cash flow stream is

$$S(h) = \sum_{i=1}^{N} P_i(h) \cdot C_i \tag{11}$$

The change in the value can be approximated in response to a small change in the random variable h by

$$\Delta S \approx \Delta h \cdot \sum_{i=1}^{N} P_i'(0) \cdot C_i = -\Delta h \cdot \sum_{i=1}^{N} P_i(0) \cdot C_i \cdot u(t_i) \cdot t_i \tag{12}$$

Traditional immunization requires choosing the asset cash flows for a given liability stream so that the linear approximation of $\Delta S = 0$. For the parallel shift model $u(s)$ can be set equal to a constant k, and then

$$\frac{\Delta S}{S} \approx -\Delta h \cdot k \cdot D_{FW} \tag{13}$$

where

$$D_{FW} = \frac{1}{S(0)} \cdot \sum_{i=1}^{N} P_i(0) \cdot C_i \cdot t_i \tag{14}$$

where D_{FW} is the Fisher-Weil duration of the value. Immunization for the parallel shift model requires choosing cash flows so that $D_{FW} = 0$.

2.3.2. Multiple direction models

If spot rate changes at different maturities are not perfectly correlated, additional factors must be added to the model.

$r_0(s)$ is initial spot rate is given by $r_0(s)$

$x(s)$ is the spot rate shift $r(s) - r_0(s)$ at date s.

$$x(s) = \sum_{k=1}^{K} u_k(s) \cdot h_k \tag{15}$$

describes now the spot rate change.

It is supposed that N cash flows are to be paid or received at the dates $t_1, \ldots, t_N$. Then $x(s)$ and $u_1(s), \ldots, u_K(s)$ can be expressed as N-dimensional column vectors. Let $X = [x(t_1), \ldots, x(t_N)]^T$, $U_1 = [u_1(t_1), \ldots, u_1(t_N)]^T$ and so forth. In vector notation, the previous equation becomes

$$X = \sum_{k=1}^{K} h_k \cdot U_k \tag{16}$$

without loss of generality it can be assumed that $U_1, \ldots, U_K$ are orthonormal. Each orthonormal vector determines a fundamental direction in which the spot rates can change. Any given shift vector X can be expressed as a linear combination of the K fundamental vectors.

For a portfolio to be immunized, it must be protected against spot rate shifts in each fundamental direction. This means that

$$\sum_{i=1}^{N} P_i(0) \cdot C_i \cdot u_k(t_i) \cdot t_i = 0 \qquad \text{for } k = 1, \ldots, K \tag{17}$$

Unlike the single-direction model, the multiple-direction model allows the shape of the spot rate curve to change in many different ways. The problem is choosing the fundamental set of direction vectors. This is done using PCA with an objective to find a small family of fundamental directions that approximately describe the history of spot rate changes.

$U_1, \ldots, U_K$ are determined using PCA. The set of fundamental directions is determined from the sample covariance matrix of spot rate changes. The set of orthonormal vectors $U_1, \ldots, U_K$ is the first K eigenvectors of the covariance matrix ranked by the corresponding eigenvalue.

After U_1 is estimated, the duration of the value can be defined as

$$D_S = \frac{\sqrt{N}}{S(0)} \cdot \sum_{i=1}^{N} P_i(0) \cdot C_i \cdot u_1(t_i) \cdot t_i \tag{18}$$

where N equals the number of elements in the column vector U_I. This definition is similar to Fisher-Weil definition, except that each date s is scaled by $\sqrt{N}$ to assure that if each component of the normalized vector U_I has the same value, the Fisher-Weil duration is obtained. Immunization against spot rate changes against U_I requires setting $D_S = 0$.

Extending the single-factor approach the duration of the surplus value can be defined for each direction $k = 1$ to K as

$$D_S = \frac{\sqrt{N}}{S(0)} \cdot \sum_{i=1}^{N} P_i(0) \cdot C_i \cdot u_k(t_i) \cdot t_i \tag{19}$$

(Barber and Copper, 1996)

2.4. Combination hedge model

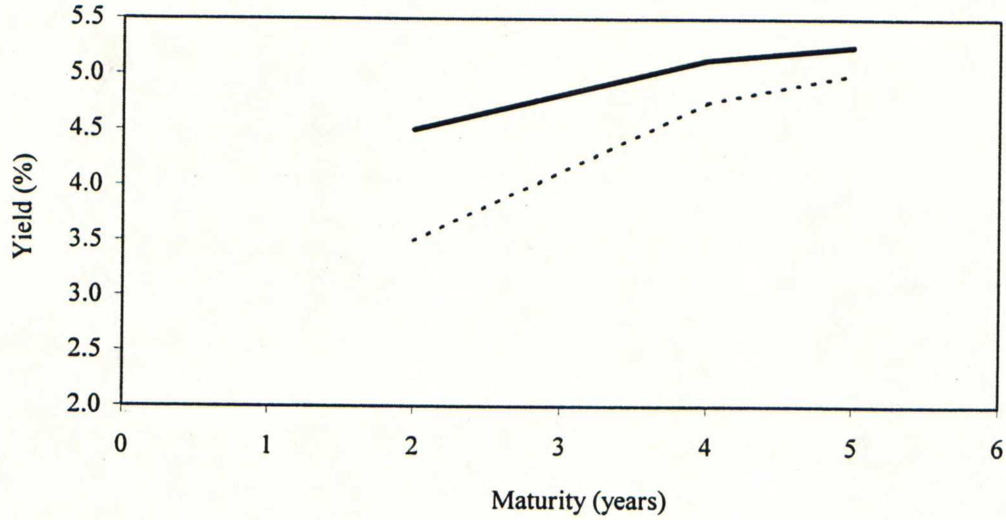
Leschhorn (2001) presents a method to evaluate and regulate the risk exposure of default-free bonds. The method generalizes the traditional duration concept by taking nonparallel changes in the term structure of interest rates into account. He calls the model combination hedge, and it is presented below. He also shows how to replicate a diversified bond portfolio by using the model and a few standard hedging instruments, and these ideas are discussed further in the next chapter. He does not present comprehensive empirical tests to back up his model.

The movement of the yield curve from time t_0 to t_1 is triggered by the change in a yield of two standard hedging instruments, in practice benchmark bonds or fixed income futures. A segment on the yield curve is described with a function

$$Y(t,T) = a(t) + b(t)F(T) \tag{20}$$

Figure 2: Example of a yield curve movement in a combination hedge model

The figure illustrates a yield curve movement in a combination hedge model. The dashed line describes the former yield curve and the thick line the new yield curve.



where *a(t)* describes the parallel shift of the yield curve, *b(t)* is a change in the slope of the yield curve and *F(T)* is an arbitrary function given by the actual shape of the yield curve. In figure 2 a change in a yield curve is illustrated. In the example maturities of the standard hedging instruments are two years (T_A) and five years (T_B). The former yield curve at t_0 has the yield of $Y_A = 3.5\%$ and $Y_B = 5\%$. It is moved to the new curve at time t_1 by the change of the yield $Y_A' = 4.5\%$ and $Y_B' = 5.25\%$.

The flattening of the curve is assumed to happen homogenously between two to five year sectors. The flattening can be calculated as follows:

$$\frac{Y_B'-Y_A'}{Y_B - Y_A} = 1 + ds \tag{21}$$

where *ds* is a small change in the slope of the yield curve in the considered segment. In figure 2 the slope decreases by 50%, so $ds = -0.5$. The ratio does not depend on the maturities or the shape of the yield curve, as long as it is monotonic. In the example, the new yield at the intermediate four-year point on the curve can be calculated as follows:

$$\frac{Y_B'-Y_A'}{Y_B-Y_A}=1+ds=\frac{Y_1'-Y_A'}{Y_1-Y_A}\Leftrightarrow 1-0.5=\frac{Y_1'-4.5\%}{4.75\%-3.5\%}\Leftrightarrow Y_1'=5.125\% \tag{22}$$

The parallel yield-curve shift component in the model is

$$Y_A'-Y_A=dY_A \tag{23}$$

The yield change in some intermediate point between e.g. two-year and five-year segment on the yield curve can be calculated directly from these equations:

$$Y_1'-Y_1=dY_A+ds(Y_1-Y_A) \tag{24}$$

The yield change of an arbitrary point between maturities of the standard hedging instruments T_A and T_B can be expressed by using two parameters, the parallel yield curve shift dY_A and the change of slope ds. Because the homogenous movement assumption concerns only a segment on a yield curve, the assumption seems reasonable. (Leschhorn, 2001)

2.5. Risk-point method

The risk-point method is presented by Dattatreya and Fabozzi (1995) as a method of controlling yield curve risk. The starting point of their approach is that the value of an asset of a given maturity might react to changes in rates of another maturity. As an example, the authors point out that for example a ten-year coupon bond pays coupons every year throughout its life. Because the value of a bond is simply the sum of the present values of the individual cash flows, it stands to reason that the value of the ten-year bond could be influenced by rate changes not only in the ten-year maturity but also in all shorter maturities representing the cash flows. The solution to this problem is to break down each asset into its cash flow components. Then the individual cash flows can be grouped into maturity buckets and the buckets' price sensitivity can be examined. The risk-point method takes a step further and attempts to integrate risk management and security valuation by providing risk measures of the bonds to be hedged relative to the hedge instruments.

An essential part of the risk-point method is a model that values the assets relative to the prices of the hedge instruments chosen. The relative valuation procedure is performed as follows: The yield and therefore the price of one of the hedge instruments is changed by a small amount, in the model by 1 basis point, and then the zero yields are recalculated. The value of the portfolio to be hedged can now be calculated again using the new zero yields as discount rates. The change in the value of the bond to be hedged relative to the change in the yield of the hedge gives the risk point relative to that hedge instrument. The method gives the amount of the hedge instrument needed for hedging by equating the risk point of the hedge to the risk point of the portfolio. The procedure of changing the yields of the hedge instruments, calculating the zero rates again and revaluing the cash flows of the bond to be hedged is repeated for all the hedge instruments chosen. It seems that the hedging performance of the risk-point method has not been statistically tested before. (Dattatreya and Fabozzi, 1995)

3. Fixed income futures contracts

Fabozzi (1998) defines a futures contract as follows: a futures contract is an agreement between a buyer (seller) and an established exchange in which the buyer (seller) agrees to take (make) delivery of something at a specified price at the end of a designated period of time.

The financial futures market permits the risk of existing and planned financial positions to be shifted to participants most willing to accept risk, that is, to participants who will charge the least for risk bearing. Furthermore, transaction costs are relatively small and the margin required to take a position is simply good faith money, which may be in the form of e.g. T-Bills. Financial futures markets are therefore an economical way to shift risk. (Hilliard, 1988)

The participant is "long" in the futures contract if he/she agrees to receive the security at the settlement date in exchange for the price agreed on today. If the participant is to deliver the security, he/she is "short". Long futures position realizes a gain when the futures price increases and a short position when the futures price decreases. Bond futures price increases when interest rates fall and decreases when interest rates rise. When fixed income futures are used for hedging a long bond position, a futures contract must be sold. This short position gains value when interest rates rise.

Once the futures contract stops trading it will become transferred into real financial asset through a delivery mechanism. The prospect of holding a contract through to delivery creates the fundamental connection between cash and futures markets. If an institution holds a short futures position when the contract stops trading, it has to deliver one of the bonds in the deliverable basket to the long, and on the contrary, if the institution holds a long futures position, it has to pay the price of the bond that is delivered to it. However, it is worth noting that entering into a futures position is not necessarily based on intention to deliver, or to take delivery of the underlying security. In the event of a price increase in the futures contract an original buyer of a futures contract is able to realize a profit by simply selling an equal number of contracts to those originally bought. The reverse applies to short position. In practice, majority of the contracts are closed by reversing positions before the maturity. Sundaresan (1997) notes that in the US Treasury bond and Treasury note futures markets more than 90% of the contracts are settled by offset. The transactions are facilitated through a third party called the clearinghouse. The clearinghouse, typically operated by the futures exchange, does not take positions in the market. Instead, the clearinghouse guarantees the performance of both sides of the contract.

Marking-to-market is an important feature of futures contracts. Each day, contracts are effectively closed out and rewritten at the new contract price. The participant's account is settled in cash according to the change in value of the contract from the previous day. A positive change in cash value of the position can be invested at market rates. Negative changes reduce the interest-bearing principal in the account.

3.1. Pricing of futures contracts

Hull (2002, pp.115) states that an exact theoretical futures price for Treasury bond contract is difficult to determine because the short party's options concerned with the timing of delivery and choice of the bond that is delivered cannot be easily valued. If it is assumed that the cheapest to deliver bond and the delivery date are known, the Treasury bond futures contract is a futures contract on a security providing the holder with known income. The futures price F_0 is then related to the spot price S_0 by

$$F_0 = (S_0 - I) \cdot e^{rT} \tag{25}$$

where I is the present value of the coupons during the life of the futures contract, T is the time until the futures contract matures, and r is the risk-free interest rate applicable to the time period T.

One feature that differentiates the forward and futures contracts is the marking-to-market feature in futures. When interest rates vary unpredictably, as they do in the real world, forward and futures prices are in theory no longer the same, if the underlying security is correlated with interest rates. When spot price increases, an investor with long futures position makes immediate gain because of daily settlement. Positive correlation of spot price and interest rates means that interest rates had also likely increased. The gain is therefore likely to be invested in higher than average interest rate. Similarly, when underlying spot rate decreases, this tends to be financed with lower than average interest rates. Investor holding forward contract is not affected in this way by interest rate movements and therefore long futures contract is more attractive than long forward contract. If the spot price is strongly positively correlated with interest rates, futures price tends to be higher than forward price. If the spot price is strongly negatively correlated with interest rates forward price tend to be higher. In practice the theoretical differences between forward and futures prices for contracts that last only a few months are in most circumstances sufficiently small to be ignored, since there are a number of factors not reflected in theoretical models that may cause forward and futures prices to be different, including taxation, transaction costs and treatment of margins. In long maturity contracts the difference can become large enough that it should be taken into account. In this case, a convexity adjustment can be made to convert futures rates to forward rates, but this is not necessary for contracts that are shorter than a year. More on this can be found in Hull (2002, pp. 52 and 117).

Cox, Ingersoll and Ross (1981) and Richard and Sundaresan (1981) demonstrate that because of marking-to-market feature, futures contracts are not the same as forward contracts when interest rates are stochastic. Figlewski (1983) states however that although the theoretical importance of marking-to-market has been discussed, efforts to determine the economic significance of this factor suggests that it is rather small (Elton, Gruber, Rentzler, 1983)

In addition to marking-to-market, bond futures contracts have also another feature that affects to their pricing. The equation 25 describes essentially the price of a bond forward contract, since it was assumed that only one bond was eligible for delivery and that the delivery date was known in advance. However, since the short futures position holds delivery options that are valuable, the price of the futures contract has to be lower than the price of a similar forward contract. The difference between a cash bond price and the futures price is called basis and it is studied in a later section.

3.2. Deliverable bonds and the conversion factor system

The delivery process for the bond futures contract makes the contract interesting. For example with the German government bond futures traded at Eurex exchange, at the delivery month, in March, June, September or December, the seller of the futures contract is required to deliver to the buyer €100 000 par value of a notional 6% coupon ten-year, five-year or two-year German government bond. Since the deliverable bonds will have slightly different maturities and different coupons, their prices will not be equal. The problem is solved using conversion factors for determining the price of each acceptable deliverable issue. Functionally, the conversion factor represents the set of prices that would prevail in the cash market if all bonds in the basket were trading at a yield equivalent to the contract's notional coupon. The underlying premise is that the futures price must be adjusted upwards, if the bond that is delivered has a coupon in excess of 6%, and downwards if the delivered bond has a coupon less than 6%. The conversion factor is determined by Eurex before the futures contract begins trading and it is constant throughout the trading period of the contract. The exact formula for the calculation of the conversion factor for the German government bond futures at Eurex is provided in Appendix B. The price the buyer of a futures contract pays to the seller at the delivery is determined as follows:

$$\textit{Price to be paid} = \text{€}100\ 000 * \textit{Futures settlement price} * \textit{Conversion factor} + \textit{Accrued interest of a bond.} \qquad (26)$$

3.2.1. Cheapest to deliver bonds

At delivery a conversion factor is used to help calculate the final delivery price for the bond. Essentially the conversion factor generates a price at which a bond would trade if its yield would equal the futures contract's notional coupon on delivery day. One of the assumptions made in the conversion factor formula is that the yield curve is flat on the time of delivery, and what is more, is at the same level as that of the futures contract's notional coupon. Based on this assumption the bonds in the delivery basket should all be equally deliverable. Of course, this does not truly reflect the reality. As a result, the implied discounting at the notional coupon level generally does not reflect the true yield curve structure. The conversion factor thus creates a bias, which promotes certain bonds for delivery above others.

This bias is caused by the incorrect discount rate, which equals the notional coupon, in the case of Eurex-traded bond futures contracts 6%, implied by the way the conversion factor is calculated. When market yields are below 6%, all eligible bonds are undervalued in the calculation of the delivery price. This effect is least significant for bonds with low duration, as these bonds are less sensitive to variations of the discount rate. As stated above, this effect is reversed for yields above 6%. (Eurex Communications, 2003)

Therefore, in spite of the conversion factor system, there is always a bond that is cheapest to deliver (CTD). In selecting the issue to be delivered, the short will select from among all the deliverable issues the one that will give the largest rate of return for a cash and carry trade. A cash and carry trade is one in which a cash bond which is acceptable for delivery is purchased and a futures contract is sold. A rate of return can be calculated for this trade. This rate is called implied repo rate. The cheapest to deliver issue is that acceptable issue that has largest implied repo rate.

The CTD-bond often changes over the life of the contract. This happens when yield changes are sufficiently large. It can be determined, which acceptable issue would become the CTD-bond with predetermined yield changes. The ratio $\frac{P_i}{CF_i}$, where P_i is the cash bond price and CF_i is the conversion factor, is known as zero basis futures price, and it is used to determine the cheapest bond to be delivered. The lower the ratio, the cheaper the bond is to be delivered. As stated earlier, an important yield level is the notional coupon level of the futures contract. For bond yields lower than 6% low duration bonds are cheaper to deliver and for yields above 6%, high duration bonds become cheaper to deliver. Earlier was stated the technical reason, why the notional coupon creates a limit that changes the behavior of the attractiveness of the deliverable bonds. The economic reasoning behind this phenomenon is as follows: As rates fall, all bonds appreciate in price, but low-coupon, long-maturity bonds tend to become relatively more expensive and therefore it is cheaper to deliver bonds with short duration. Conversely, as the rates go up all bonds become cheap, but the low-coupon, long-maturity bonds tend to become cheaper than the high-coupon, short-maturity bonds. As a consequence, low-coupon, long-maturity bonds are delivered during periods of high interest rates. (Sundaresan, 1997)

The existence of a cheapest to deliver bond has some important implications that affect strongly to the hedging with futures contracts. Assuming the identity of the CTD-bond is known to market participants, the bond futures contract should trade as if it is a contract on the cheapest to deliver bond and therefore the contract should reflect the risk characteristics of the cheapest to deliver bond projected as of the maturity date of the futures contract (Rendleman, 1999). This relationship between the risk characteristics of the CTD-bond and the futures contract is used extensively in the empirical section of this study.

3.2.2. Embedded delivery options

In addition to the choice of which acceptable bond issue to deliver, sometimes referred to as the quality option, the short position has also two more options granted. Sundaresan (1997) uses US Treasury futures market as an example and calls the options available as the end-of-the-month option and the wild-card option. The Treasury futures contract stops trading seven days before the last possible day to deliver the bond. Therefore the futures price is fixed, but the price fluctuates in the underlying Treasury bond market. End-of-the-month option gives the short a right to decide when to deliver during these days. However, when yield curve is not inverted, the short will usually deliver the bond as late as possible due to the positive carry. The third option is called the wild-card option. It is a right to give notice of intent to deliver later during the same day than the futures settlement price has been fixed. For example in the US, the Treasury bond futures market closes for the day at 2:00 PM Chicago time. However, the clearing house of the Chicago Board of Trade (CBOT) accepts delivery until 8:00 PM. Therefore the short has each day before the contract ceases trading in the delivery month a put option, whose strike price is set at 2:00 PM and which expires at 8:00 PM. (Sundaresan, 1997)

All the embedded options belong to the seller of the futures contract, so it could be assumed that they have also implications to the pricing of futures contracts. Broadie and Sundaresan

(1992) examine whether futures prices are bid down by the value of the options present in the futures contracts. Using a single factor model they conclude that delivery options are important and that they affect the futures price. From an empirical standpoint, the difference between the cash price and the adjusted futures price (invoice price) as of the first delivery date gives the market value of the wild-card and the end-of-the-month options. They find that the discounts are of the order of 9 to 16 ticks for US Treasury bonds and 13 ticks for US Treasury notes on the first delivery date.

3.3. Basis

The basis refers to the difference between the futures price and the price of the cash market instrument. When these prices are perfectly correlated over time, the hedge will remove all risk. However, at times the basis will fluctuate unpredictably, leaving the hedged portfolio subject to interest rate risk. Basis risk is greatest when instruments in the cash market portfolio are fundamentally different from the futures market instruments. For example, the hedge of a financial institutions portfolio of securities with a portfolio of futures instruments would have substantial basis risk while the hedge of T-bonds with T-bond futures would have minimal basis risk. (Hilliard, 1988)

Return variability increases directly with the length of the time interval considered. This means that basis risk as a fraction of total risk should decrease as the holding period is extended and hedging effectiveness should improve. Since the basis must go to zero at expiration, but is uncertain before that, it stands to reason that the closer the future is to maturity the smaller deviations from the equilibrium value will be. This suggests that hedging effectiveness may go up as the future gets close to expiration.

Sundaresan (1997) defines basis in bond futures as follows: Let P_t be the flat price of the deliverable bond, CF be its conversion factor and $H_t(s)$ be the futures price at date t for maturity at date s. Recognizing that futures contracts permit delivery on any business day of the delivery month, the s will be interpreted as the last business day of the delivery month in a market, where yield curve is upward sloping, i.e. the carry is positive. The basis B_t is defined as

$$B_t = \dot{P}_t - CF \cdot H_t(s) \tag{27}$$

If t happens to be in the delivery month, then by the no arbitrage principal, the basis must be above zero. If this were not the case, by simultaneously selling the futures contract and immediately delivering, one could lock in riskless profits.

According to Plona (1997, pp.175), the basis can be thought to consist of three components. First, it contains a price discount component representing the coupon income that the holders of the CTD-bond will earn between the current spot delivery date and the contract's delivery date. Second, it contains an additional discount to reflect the uncertainty associated with the bond that will ultimately be delivered against the contract when the delivery date does arrive. This second discount, the net basis, is often considered to be the premium of the delivery option. The third component of basis is the financing component, a reflection of the opportunity cost of bond ownership.

Different authors use slightly different notation for the same general idea of the basis decomposition. Koenigsberg and Bourtzos (1991) divide the basis into four components that are

- The carry
- The delivery option
- The swap value between the cash bond and the CTD
- Cheapness/richness of the contract

However, to determine the cheapness or richness of a contract one would need a relative value model that would provide an estimate of the correct value of the delivery option. The 'carry' term in Koenigsberg's model combines the coupon income and financing components of the Plona's presentation. The equations provided next will follow the notation of Koenigsberg and Bourtzos (1991).

Because the delivery price for a bond is futures price times the conversion factor, the futures price is less (more) sensitive than the bond price if the conversion factor is greater than (less than) one. An approximate way to compute the price sensitivity of the futures contract is to assume that it tracks the CTD-bond. Then the futures price sensitivity will equal the forward price sensitivity of the CTD-bond divided by the conversion factor of the CTD-bond. The differences between the spot and forward sensitivity are larger for a two-year bond than for a ten-year bond, because the time to the delivery month is a larger fraction of the maturity of the underlying cash instrument.

$$\textit{Futures price sensitivity} = \textit{Forward price sensitivity}_{CTD} \,/\, CF_{CTD} \tag{28}$$

The futures and cash legs of a conversion factor weighted basis position thereby have roughly the same sensitivity because the conversion factor cancels out. This is because the cumulative variation margin becomes equal to the difference between the initial and the final futures prices, since in the end, the long pays to the short the amount that equals the futures price times the conversion factor, and if the long holds exactly 1/CF futures contracts, the long will pay the amount that equals the futures price for each bond. This is the reason conversion factor weighting is used.

$$\textit{Basis}_{CTD} = \textit{Bond Price}_{CTD} - CF_{CTD} \cdot \textit{Futures price} \tag{29}$$

$$\textit{Futures price} = \textit{Bond Forward Price}_{CTD} \,/\, CF_{CTD} - \textit{Delivery option} \tag{30}$$

$$\textit{Basis}_{CTD} = \textit{Carry}_{CTD} + CF_{CTD} \cdot \textit{Delivery option} \tag{31}$$

The carry component of the basis is the difference between the cash price of the bond and its forward price, and it is a function of the spread between the finance rate and the current yield and the period for which the spread is earned. For an upwardly (downwardly) sloping yield curve the forward price is calculated to the end (beginning) of the contract month and the delivery option represents the option to deliver a bond other than the original CTD-bond some time before the end (some time after the beginning) of the month.

For non-CTD-bonds, the basis also includes a forward swap between the non-CTD-bond and the CTD-bonds:

$$Forward\ swap = CF_{non\text{-}CTD} \times [Forward_{non\text{-}CTD} / CF_{nonCTD} - Forward_{CTD}/CF_{CTD}] \quad (31)$$

Because the converted forward price is lowest for the CTD, the swap value term is positive.

3.4. Hedging bonds with bond futures

Since it seems that hedging a bond portfolio with futures contracts is not trivial at all, why is it so popular? The first reason is probably the enormous liquidity available in the futures market. In addition, Plona (1997) provides other reasons. He notes that futures, in a portfolio context, are a tool of separating the question of interest rate risk from the questions of securities selection. Securities can be selected based on long-term funding requirements, current income potential, duration, convexity or liquidity considerations. A futures position overlaid upon the portfolio will alter its exposure to the changes in the interest rate environment while leaving the underlying cash flow structures intact.

The problems in the futures hedge arise from several sources. Plona (1997) states that the delivery option can have a major effect on the performance of the hedge for two reasons. First, because its presence offers a prospect of excess returns to hedgers. Conversion factor weighting in hedging leaves a basis position exposed to changes in the value of the delivery option. This exposure arises because the futures and cash legs of the conversion factor weighted basis do not move one for one when yields change. This is due to the existence of the delivery option. When yields change, the need of short futures contracts to hedge the position changes. The second major effect on the performance of the hedge is due to the fact that the hedger is required to pay for the option to get excess returns, and the price is not trivial.

Therefore, if one truly wants to eliminate the risk of owning a bond, the very best way of hedging that bond is to sell it and invest the proceeds into the overnight repo markets. The income that can be earned from the repo market represents the best theoretical target for the bond holder who decides to hedge rather than sell the position. This alternative allows the hedger to keep the coupon income from the bond, but he will lose the amount corresponding the basis convergence[4]. (Plona, 1997)

Surprisingly, a futures hedge of a long bond portfolio can be compared to as taking a basis trading position and a view on the future behavior of the basis. A long position in basis is acquired by being long in the bond and shorting the futures contract, which is the case while hedging bond portfolios using futures contracts. For example, if a conversion factor of the bond would be 1.20, then going long 1 million of the basis would involve buying 1 million face value of bonds and selling 12 futures contracts, each with a face value of 100 thousand. Therefore, one can be long or short the basis without engaging in an explicit basis trade, and it seems necessary to study the behavior of the long basis position in order to explain deviations in the hedging performance. The long basis position will gain, which means the hedged position will produce additional returns, if (1) the short end of the yield curve decreases or (2) the CTD-bond changes due to changes in the spreads between the deliverables, a change in the overall market yield levels (up or down), or a change in the yield volatility. On the other hand, the long basis position will lose value, if the short end of the yield curve increases or the CTD-bond does not change. These factors affecting the performance of the hedge will be discussed next. (Koenigsberg and Bourtzos, 1991).

3.5. The effect of yield curve changes on the short futures position

A reshaping of the yield curve affects both the carry term and the delivery option component of the basis. The carry component of the basis is the difference between the cash price of the bond and its forward price, and it is a function of the spread between the finance rate and the current yield and the period for which the spread is earned. The current yield is defined as the coupon rate divided by the full price of the bond.

On an upward sloping yield curve the forward price is calculated to the end of the contract month and would be less than the cash price. Consequently, buying the bond and financing it to the forward rate will earn the difference between the current yield and the financing rate, if

[4] Since the futures price is lower than the cash market instrument price prior to the maturity of the futures contract, but the prices have to be equal in the maturity, the futures price will have to increase in relation to the cash bond price in order to reach the same price at maturity. This phenomenon is called basis convergence.

we assume that the cash price is fixed. If the yield curve steepens, the carry would increase as the forward price decreases relative to the cash price. Therefore the value of the basis will increase, if the position is long the basis, which is the case when bonds are hedged with the short futures contracts. When the trade is put on closer to the delivery month, the carry component is smaller because the yield differential is earned for a shorter period of time, and the basis will be smaller as cash price and futures price tend to converge.

Another important component of the basis is the delivery option. Since the seller has all the option rights, the existence of these options lowers the futures price. If the yield level changes enough so that the CTD-bond changes, it profits the long basis position. Before the CTD-bond change, the holder of the position had a CTD-bond and a short futures contract, and since after the change another bond is cheaper than the previous one, the holder of the position can sell the previous CTD-bond, buy the new cheaper one, and make a profit.

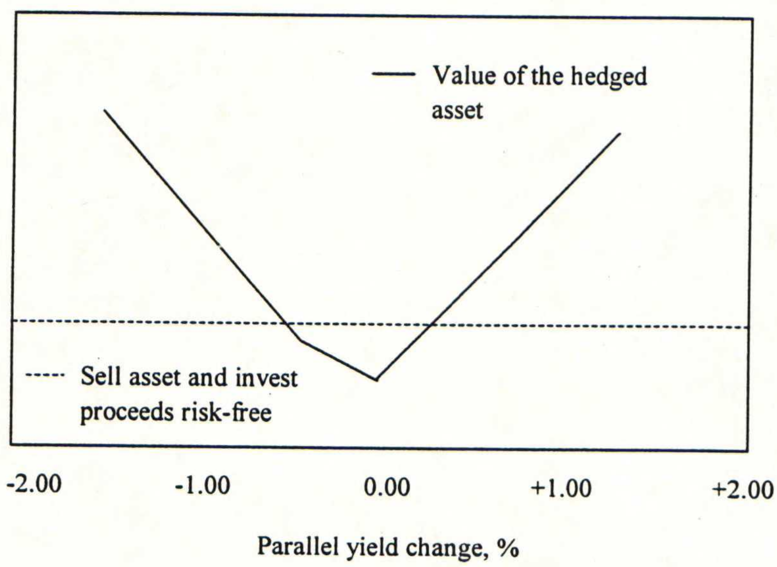
The opportunity to make a profit because of the CTD-bond change creates an option-like pattern for returns from the delivery option. Whether the basis acts like a call, a put, or a combination of them, depends on the yield level. An important yield level is the coupon rate of the notional bond that is behind the calculations of the conversion factors. With the German bond futures traded at Eurex this notional coupon rate is 6%. If the starting yield level would be exactly the 6%, the return pattern of a long position in the delivery option would look like an option straddle. This shape is caused by the following facts: as rates go down, the CTD-bond switches to a lower duration bond and the long position in the cash bond increases faster than the short futures leg of the basis. If rates go up, the CTD-bond changes to a higher duration bond and the short futures leg of the basis decreases faster than the value of the cash bond. (Koenigsberg and Bourtzos, 1991)

Figure 3 is adapted from Plona (1997) and it depicts the situation, in which a CTD-bond changes. The starting yield level (i.e. where yield change is zero) is assumed to be the notional coupon level of the futures contract, and the return pattern of the basis looks like a straddle. If the yields do not change enough, the combined position of long bonds and short futures contracts will yield less than would be available by selling the bond and investing the proceeds in the repo market. This is due to the basis convergence. If the starting yield level would be higher than the notional coupon level, the shape of the proceeds would look like a call option, and if the yield would be lower than it, the shape would look like a put option.

Figure 3: Example of the effect of CTD-bond change to returns on the hedged position

The figure describes the returns of a bond hedged with futures contracts compared with income available in repo market. The dashed line describes the level of known income that could be achieved by selling the bond and investing the money risk-free. The solid line describes the returns available, if investor decides to hedge the long position with futures contracts. The starting yield level is assumed to be the level of notional coupons in the futures contract.

Source: Plona (1997)

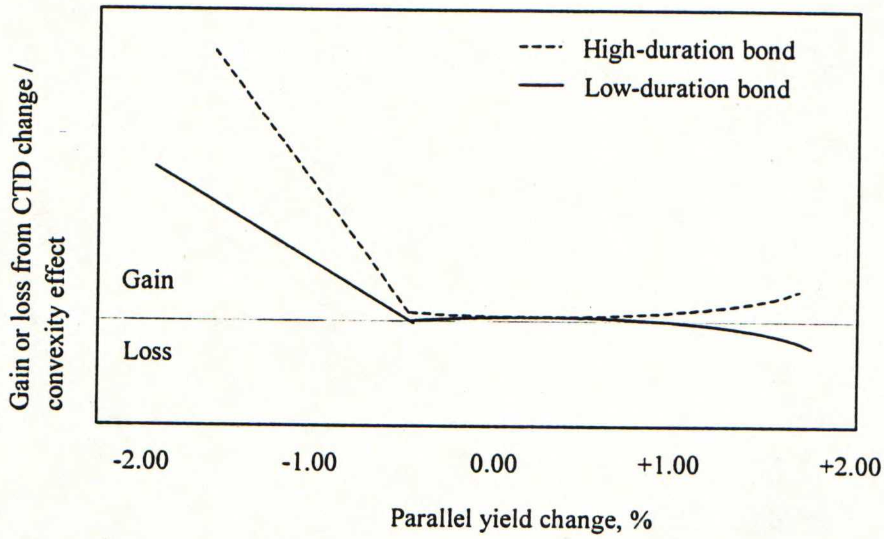


Usually the hedged bond is not the cheapest-to-deliver bond. This situation is described in figure 4, in which a hedge of two non-CTD-bonds is depicted. Another one of them has lower, and another one higher duration than the CTD-bond. The notional coupon level is reached, and the switch in which bond is cheapest to deliver occurs in the figure after the yields have decreased by 50 basis points. The figure describes two separate factors that have effect on the value of the basis. The first one is the prospect of a CTD-change. When the yields fall under the notional coupon level of the futures contract, the bond in the deliverable basket with short duration will be the new cheapest to deliver bond. In the example, this new CTD-bond has lower duration than the low-duration bond that is being hedged. Therefore also the low-duration bond is effectively long futures contracts. Therefore, when the CTD-switch occurs, both hedge positions become bullish. The low-duration bond's hedge position is slightly less bullish, and therefore the gain is smaller than with the high-duration bond.

Figure 4: Gain or loss associated with hedging bonds that are not cheapest to deliver on the last futures trading date.

High-duration bond means a bond, which duration is higher than the CTD-bond's duration. Low-duration bond means a bond, which duration is lower than the CTD-bond's duration. The notional coupon rate is assumed to be 0.50% lower than the current yield level.

Source: Plona (1997)



Another factor affecting the performance of the hedge is convexity. Although the cheapest to deliver bond remains the same, the performance of the hedged position can fluctuate. This can be observed in the figure 4 when yields begin to increase. Although the hedge ratio calculation takes into account the duration differences of the bond that is being hedged and the CTD-bond, it does not take into account that because of the differences in the hedge ratios, the convexities of the bonds also differ, i.e. yield change causes the basis point values of the long and short positions to change at different speeds. The basis point value of all three positions will decrease as the yields increase, this is a basic property of the non-linear relationship between bond yield and price. However, the change is not equal for all the bonds. In the first case, the bond that is being hedged has higher duration and also higher convexity than the CTD-bond underlying the futures contract. After the yield increase, a high-duration position is essentially short futures contracts, which causes the position to gain when interest rates rise. The reason the position is short is that the basis point value of the CTD-bond has decreased more than the basis point value of the high-duration bond. In the hedge ratio calculation the basis point value of the bond to be hedged is divided by the basis point value of the CTD-bond. The amount of futures contracts should therefore be higher than it is, which makes the position short futures contracts. The opposite happens to the bond that has lower duration than the CTD-bond. The convexity effect does not create major problems unless the yield changes are quite large.

The yield volatility also affects to the value of the delivery option. As with all options, the increase in volatility increases the value. If the uncertainty of the rates increases, so does the chance that the CTD-bond will change. This increased uncertainty is reflected as an increased value of the delivery option.

Several factors affect to the value of the basis and therefore the value of the hedged position that consists of long bonds and short futures contracts. A summary of the effects is presented on table 1, which describes the effects that a change in interest rates or a yield curve shape can have on the long basis position. In a portfolio of long bonds and short futures contracts the implicit basis position is long. Although the previous discussion has highlighted the situation, in which the CTD-bond changes because the yield level crosses the notional coupon level, this does not mean that the cheapest to deliver bond could not change otherwise. When yield levels change, the CTD-bond can always change, but determining when the change will occur, and which bond will be the new CTD-bond always requires careful analysis, and depends on the situation. More information can be obtained from Plona (1997).

Table 1: Summary of the effects of yield curve changes on the long basis position

The portfolio consists of a long position in government bonds and a short position in futures contracts on the bonds. The table describes the effects of yield curve changes to the value of the basis in this position.

Yield curve change	**Basis value**
Carry	
Yield curve steepening	Increases
Yield curve flattening	Decreases
Delivery option	
Yield level change (prospect of CTD change)	Increases
Yield volatility increase	Increases
Yield volatility decrease	Decreases
Convexity effects	
Large yield change (if CTD duration lower)	Increases
Large yield change (if CTD duration higher)	Decreases

A change in the CTD-bond may produce large gains in the hedged position. However, this opportunity for a gain does not come for free. The possible gains from CTD-switches occur instantaneously, but the futures hedge positions are usually held for a longer period of time, which makes the position exposed to the futures contract's basis convergence. The basis convergence dramatically reduces the potential income available. In addition, the position suffers from negative convexity, if the duration of the bond that is being hedged is lower than the duration of the CTD-bond. If the hedge would be held until delivery, the return on this position would be lower than the money market rate. Therefore, the decision to hedge a bond with futures, if informed, is the decision to sacrifice risk-free income with the aim either of capturing a gain through a cheapest-to-deliver bond switch or by avoiding transaction costs or adverse tax consequences. (Plona, 1997, pp. 178)

Summary of the factors causing mismatch to the futures hedges:

- Cheapest-to-deliver bond and the asset that is being hedged will not move completely in parallel

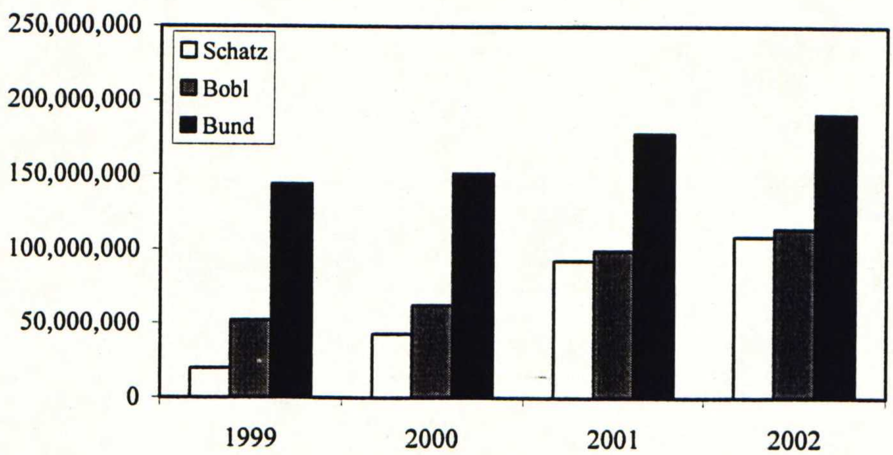
- Since hedges are not kept until the delivery, the size of the basis at the moment the hedge is rolled over to the next contract is unknown. Most likely it has converged from the original amount, but it is still not zero.
- In practice CTD-bonds change during the hedges.
- Convexity differences between the bond that is being hedged and the CTD-bond cause mismatch to the behavior of the hedged bond and the futures position, especially when the yield changes are significant. The position suffers from negative convexity, if the duration of the bond that is being hedged is lower than the duration of the CTD-bond

3.6. The German fixed income futures market

Trading of fixed income futures has grown strongly in Eurex exchange during the first four years these contracts have been nominated in euros. This can be seen from figure 5. For example the number of traded Schatz futures contracts has grown 450% from 1999 to 2002. The number of traded Bobl contracts has increased over 120%, and the number of traded Bund contracts over 30%. However, only a small portion of the traded contracts lead to the delivery of the underlying bond.

Figure 5: Number of traded fixed income futures contracts in Eurex Exchange between years 1999 - 2002

Source: Eurex



The underlying instruments for the Eurex-traded ten-year Bund, five-year Bobl and two-year Schatz futures contracts are ten-year, five-year and two-year notional 6% coupon German government bonds. Only by pure chance there would be such a bond traded in the markets when the futures contract matures. Therefore the Eurex exchange has defined delivery rules that allow one of several German bonds to be delivered. The choice of which bond issue to deliver from among those in the pool that may be delivered is given to the seller of the futures contract. The most important characteristics of the futures contracts are listed on a table 2.

Table 2: The contract specifications for Bund, Bobl, and Schatz futures contracts

Source: Eurex

	Schatz	Bobl	Bund
Nominal value of the contract	€ 100,000	€ 100,000	€ 100,000
Maturity range of deliverable bonds	1.75-2.25 years	4.5-5.5 years	8.5-10.5 years
Minimum issue size of deliverable bonds	€ 2 bn	€ 2 bn	€ 2 bn
Minimum price movement	0.01% = EUR 10	0.01% = EUR 10	0.01% = EUR 10
Delivery day	10th day of March, June, September and December		

4. Data and methodology

The empirical section of the thesis studies the performance of different hedge ratio estimation methods using data of German government bonds since Germany joined the European Monetary Union (EMU) at the beginning of 1999. The performance measurement of the different methods is carried out as follows: the hedge ratios are calculated each week using out-of-the-sample data, and the futures trades are executed according to these calculations. The change in value of the portfolio that is being hedged, and the gains or losses of the short futures position, are recorded weekly. The standard deviation in the value of this combined position of long bonds and short futures contracts is the first key measure used to compare the hedge ratio estimation methods. Naturally the reasoning is: the smaller the standard deviation, the better the hedge.

Another important measure that is used, is the amount of futures trades the methods involve. This is a direct method of measuring the cost of the hedges. The hypotheses are that the remaining standard deviation will be somewhat higher in portfolios hedged with duration or basic regression, but on the other hand, the costs of the hedge will be considerably lower than with portfolios hedged using PCA, combination hedge or risk-point method.

The empirical analysis in the thesis consists of four main parts. First, principal component analysis will be used to estimate factors that would explain most of the variance on the German yield curve. Second, single German government bonds will be hedged for the period of three years and five months using five different methods to calculate the hedge ratios. Third, the same analysis is conducted on portfolios that consist of all the available German bonds in the data set. Finally, I will study the behavior of the remaining returns of the hedged portfolios using regression.

To perform the comparison of the methods, the following data are needed:

- Data for the underlying portfolio that is subject to the interest rate risk. In this study I hedge (a) single coupon bonds and (b) a portfolio consisting of these bonds. The data needed for the bonds are from Bloomberg. The data are daily closing values at 19:00 CET. The data set covers the period 01/01/1999 – 05/31/2002.

- Daily historical futures price data for Schatz (two years maturity), Bobl (five years maturity) and Bund (ten years maturity) futures contracts. In addition, data of the cheapest to deliver bonds on each day, including their yields, prices and modified durations are needed. Historical futures prices and the corresponding cheapest to deliver bond data are from Bloomberg using Bloomberg generic mid prices that are composed of quotes from several large investment banks. The data are daily closing values at 19:00 CET.

- Zero yields estimated using the Nelson-Siegel method. The estimation is performed using Nordea Analytics software. The daily German zero yields are estimated for the period 01/01/1996 – 05/31/2002. The estimation software uses daily Reuters quotes at 16:30 CET. However, for reasons explained later, zero yields estimated with bootstrapping are used with the risk-point method.

In this chapter I will also present some spreadsheet examples of how the more complex methods, namely the principal component method and the risk-point method, were implemented in this study. I chose the date of the examples to be the 27th of May, 2002.

The table 3 presents the German bonds that are being hedged in this study. The data set includes all the bonds that were issued by German government, whose maturity were originally ten years, which were available during the estimation period of 01/01/1999 – 05/31/2002, and whose time to maturity exceeded one year at the end of the estimation period.

Table 3: Bonds that are being hedged in the study

Data includes all the bonds that were issued as ten-year bonds by the German government and whose maturity were longer than one year for the period 01/01/1999 - 05/31/2002

Maturity	Coupon	Time to maturity on 01/01/1999	Time to maturity on 05/31/2002
07/15/2003	6.5	4.54	1.12
09/15/2003	6	4.71	1.29
07/15/2004	6.75	5.54	2.13
11/11/2004	7.5	5.87	2.45
01/03/2005	7.375	6.01	2.60
05/12/2005	6.875	6.36	2.95
10/14/2005	6.5	6.79	3.38
01/05/2006	6	7.02	3.60
02/16/2006	6	7.13	3.72
04/26/2006	6.25	7.32	3.91
01/04/2007	6	8.01	4.60
07/04/2007	6	8.51	5.10
01/04/2008	5.25	9.01	5.60
07/04/2008	4.125	9.51	6.10
07/04/2008	4.75	9.51	6.10

To hedge using principal component analysis or the risk-point method, also zero-coupon yields are needed. As a zero-coupon bond does not have coupons, there is only one cash flow during the life of the bond. This makes the valuation of the bond simpler since there is no risk of the interest rate with which the coupons can be reinvested. However, zero-coupon yield data are not publicly available since the bond yields that can be observed in the markets are usually quotes for coupon bonds. In addition, one should remember that the maturity of a benchmark bond is exactly ten years only on one day during its life. In addition to the maturity problem, bonds may 'trade special' in repo markets or the size of the coupon may affect to the valuation of the bond. In order to get more exact ten-year yield quotes for each day in the estimation period, the zero yield curves must be estimated for each day using statistical techniques and the available coupon bond data. There are several different methods available for this estimation procedure.

4.1. Zero yield curve estimation

Yield curve estimation methods have been a major area of research in central banks. The competing estimation methods in the literature include Svensson (1994) model, which is also sometimes referred to as extended Nelson-Siegel model due to its quite similar functional form, and a more parsimonious functional form of Nelson-Siegel (1987) based model. In addition to these parametric models, there are spline-based methods presented e.g. by McCulloch (1975) and Fisher, Nychka and Zervos (1995). However, as Bolder and Streliski (1999) note, the different methods can provide surprisingly different shapes for the yield curve and the selection of the model depends on its final use.

Anderson and Sleath (2001) compare alternative yield curve estimation methods using three criteria. First, the technique should give smooth forward curves rather than trying to fit every data point if the purpose is not to price a security. Second, the model should be flexible enough to capture movements in the term structure. More flexibility is needed especially in the short end of the yield curve where expectations are better informed and more subject to revision as news reach the market than at the longer end. Third, the technique should produce stable yield curves i.e. estimates at any particular maturity should be stable in the sense that small changes in the data at one maturity do not have a disproportionate effect on forward rates at other maturities.

Anderson and Sleath (2001) compare the parametric methods to the spline-based methods. They state that spline-based methods allow much higher degree of flexibility than either of the two parametric models, which is the principal advantage of them. Specifically, individual curve segments can move almost independently of each other so that separate regions of the curve are less affected by movements in nearby areas. This is in contrast to the parametric forms for which a change in the data at any one point can affect the entire curve, as estimates at any maturity are a function of all the parameters to be estimated. Between parametric methods, the authors conclude that Nelson-Siegel model appears to be much more stable than the Svensson technique.

Bolder and Streliski (1999) compare the parametric Svensson model and Nelson-Siegel (NS) model to Canadian central bank's previously used Super-Bell model, which had been much easier to estimate using basic ordinary least squares regression on the observed yields data and then using bootstrapping to estimate the yields of desired maturities (e.g. two-year, five-year, ten-year yields). In bootstrapping the yield curve is assumed to be piecewise linear, which leads to non-smooth yield curve. However, bootstrapping is quite easy to implement compared to the other methods that result a smooth yield curve and, maybe more importantly, it gives quite accurate estimates for the yields, which is important e.g. when these yields are used in pricing of securities. The authors give several factors that support the use of a Nelson-Siegel based model to construct the zero yield curve. First, NS produces a forward curve that can be used as an approximation of aggregate expectations for future interest rate movements. Second, the model focuses on actual cash flows of the underlying coupon bonds instead of yield to maturity, which are subject to a number of shortcomings (see Bolder and Streliski, 1999). Third, the functional form of NS is capable of taking into account several possible shapes in the yield curve and last, the continuous zero-coupon curve avoids the need to use other models to interpolate between intermediate points. Bolder and Streliski (1999) point out the common statement that Nelson-Siegel based models are not especially useful in pricing securities.

The estimated zero yield curves are always more or less inaccurate i.e. the actual bond yields do not lie exactly on the estimated yield curve. On the other hand, the methods that reflect better the actual quoted yields tend to produce yield curves that are less smooth, which often results theoretical problems with the estimated yields (e.g. negative forward rates). Therefore, the selection of the estimation method is a compromise between accuracy and smoothness.

4.2. Regression-based hedge

The hedge ratio is the estimated beta coefficient from the regression of futures price changes on the changes of the portfolio returns. The regression is of the form

$$y_i = a + bx_i + e_i \tag{32}$$

Where $y_i = \ln(\frac{P_1}{P_0})$

$x_i = \ln(\frac{F_1}{F_0})$

P_1 = bond price at time 1

P_0 = bond price at time 0

F_1 = futures price at time 1

F_0 = futures price at time 0

The regression is performed once a week using daily observations from rolling three-month period prior to the hedge. The daily observations are chosen instead of weekly observations in order to get enough data for the regression. If weekly observations were used, the estimation window would have to be too long in order to achieve enough data points. The relation between bond and futures changes constantly as time goes by and the maturity of the bond shortens. In the regression, only one hedge instrument is used at a time due to serious multicollinearity that would emerge if multiple regression were used. The futures contract, which maturity is nearest to that of the bond to be hedged is used. However, a constantly changing CTD-bond could cause major negative effects to the simulation procedure, because the new CTD-bond has naturally different maturity than the previous one, which could cause the hedging instrument to change. This would lead to unnecessarily large transaction costs that would not reflect the reality. Therefore, the futures contracts are used as follows: if the time to maturity of the bond to be hedged is less than 3 years, Schatz contracts are used. If the modified duration is between 3 and 7 years, Bobl contracts are used, and for the bonds whose modified duration is longer than 7 years Bund contracts are used.

4.3. Duration hedge

Modified duration is related to a percentage price change of a bond. However, for two bonds with the same modified duration the dollar price change will not be the same if the prices are not the same. In the framework of hedging, the proper risk measure is dollar duration, which is defined as modified duration times the bond's price. It can be calculated using bond's basis point value. The bond's interest rate sensitivity is presented in terms of the change in its price associated with a single basis point[5] change in yield. Basis point value is a concept that describes the change in the bond position value (in euros) given a change in interest rates. A percentage change in a bond price can be expressed using modified duration as follows:

$$\frac{\Delta P}{P} = -MD \cdot \Delta y \tag{33}$$

The bond price change using the concept of basis point value (*BPV*) is simply the modified duration times the bond's dirty price, i.e. the bond's flat price and the accrued coupons:

$$\Delta P \approx MD \cdot P \cdot \Delta y = -BPV \cdot \Delta y \tag{34}$$

A futures contract does not have fixed contractual cash flows so it does not have duration either. However, the risk characteristics of the cheapest to deliver bond can be used.

Rendleman (1999) makes a survey of the formulas that different finance textbooks present to calculate the appropriate hedge ratio, when Treasury bonds are hedged with duration. He states that there are four different methods presented, and each of them leads to a different result. After that he derives the appropriate hedge ratio to be used when bonds are hedged with futures contracts using duration. In this study, this Rendleman's hedge ratio is applied. It is also equal to the most common formula provided in the textbooks. Rendleman (1999) notes that the use of modified duration is appropriate when hedging against a potential change in semiannually or annually compounded yields. Non-modified duration is the appropriate measure when hedging against a potential change in continuously compounded yields. Because Rendleman (1999) derives the hedge ratio he uses for simplicity continuously compounded yields. Since in this study the yields are compounded annually, the duration is replaced by modified duration.

The hedge ratios presented by the different models tend to be based on the concept of the basis point values, but the differences between the models arise from the fact at which time

[5] One basis point equals 0.01%.

point these values should be obtained. The Rendleman's model calculates the amount of needed futures contracts as follows:

$$N = \frac{D_S \cdot (P_S + A_S)}{D^*_{CTD} \cdot (P^*_{CTD} + A^*_{CTD})} \cdot CF \tag{35}$$

where N = number of futures contracts to be sold short

D_S = modified duration of the bond to be hedged at the time of the hedge

D^*_{CTD} = modified duration of the CTD-bond at the maturity of the futures contract

P_S = price of the bond to be hedged at the time of the hedge

A_S = accrued interest of the bond to be hedged at thevtime of the hedge

P^*_{CTD} = price of the bond to be hedged at the time of the hedge

A^*_{CTD} = accrued interest of the bond to be hedged at the time of the hedge

CF = conversion factor of the CTD-bond

Futures maturity is on an upward-sloping yield curve the last day of the delivery month. The derivation of the basis point value of the futures contract requires some further assumptions. It has to be assumed that the cheapest to deliver bond remains the same over the life of the hedge. The assumption of unchanged CTD bond does not hold in practice. The CTD-bond can change constantly during the contract period.

It should be noted that in the duration method only one hedging instrument is used at the time. The hedging instrument is chosen similarly than with the regression hedge method. If the modified duration of the bond to be hedged is less than 3 years, Schatz contracts are used, if it is 3-7 years Bobl contracts are used and if it is longer than 7 years Bund contracts are used.

4.4. Principal component analysis in hedging

Principal components are estimated from Nelson-Siegel zero-coupon yield change data. Since the purpose of this study is to compare the performance of different hedge ratio estimation methods, the estimation window of the yield changes differs from the ones in earlier studies that use PCA in hedging. In order to be able objectively compare the hedging results obtained in the beginning of the estimation period, i.e. in January 1999 with the hedging results obtained in May 2002, the PCA estimation time frame has to remain constant. In this study, new principal components are calculated each week using rolling three-year data of spot rate changes. Therefore 176 sets of principal components are estimated. In the previous studies usually just one set of principal components is used, which does not reflect the hedging situation in practice.

Since the hedging period begins at the beginning of year 1999, zero yield curve data are needed from the year 1996. The zero yield curve data include estimates from zero to ten years in 0.25 year increments so there are 40 weekly observations in the zero-coupon yield curve. First, the logarithmic weekly changes of the zero coupon rates are calculated. Then, the averages of the three-year data are deducted from the observations of each maturity sector. After the average is subtracted, the covariance matrix of the data is calculated. Using matrix algebra, eigenvectors and corresponding eigenvalues are calculated from the covariance matrix. The factor loadings for each principal component and standard deviations of factor scores are obtained from these eigenvectors. Each eigenvector is a principal component and the corresponding eigenvalue indicates how much variation in the data that component explains. The eigenvalues are sorted from largest to smallest so the first component explains the largest amount of variance. It can be assumed that the three most important principal components explain together more than 90% of the variance. The three vectors that have the largest eigenvalues are the three principal components of the covariance matrix. Table 4 presents an example of eigenvectors that represent the three principal components on 27 May 2002. Table 5 provides an example how to calculate the hedge ratios after eigenvectors and eigenvalues are obtained. The procedure is to follow the method presented by Barber and Copper (1996) and derived in the previous chapter.

Example of some of the calculations in table 5:

- $u_1(t_i)$, $u_2(t_i)$, $u_3(t_i)$ are the factor loadings for the corresponding time for the particular coupon flow, see table 3
- Exposure to U_1 is $\frac{\sqrt{40}}{102.93} \cdot 87.82 = 5.40$, from equation 18.

- Exposure to U_2 is $\frac{\sqrt{40}}{102.93} \cdot 22.17 = 1.36$
- Exposure to U_3 is $\frac{\sqrt{40}}{102.93} \cdot (-13.37) = -0.82$
- Number of short Schatz, Bobl and Bund futures contracts is calculated using simple matrix algebra. The exposures of the bond to be hedged to *U1*, *U2* and *U3* will have to equal the sum of the exposures to the components by the CTD-bond of Schatz, Bobl and Bund futures.

Table 4: Example of eigenvectors

The table presents the eigenvectors estimated on 27 May 2002 from the covariance matrix of weekly logarithmic zero-rate changes using German government bond data from the preceding three-year period. Zero-rates were estimated using Nelson-Siegel method.

	U_1	U_2	U_3	$U_1+U_2+U_3$
Explained variance	69.6%	25.6%	4.3%	99.5%

Maturity	U_1	U_2	U_3	Zero rates
0.25	-0.0278	-0.5909	-0.4496	3.52%
0.50	0.0360	-0.4709	-0.1464	3.65%
0.75	0.0797	-0.3748	0.0165	3.78%
1.00	0.1117	-0.2979	0.1096	3.89%
1.25	0.1357	-0.2357	0.1632	3.99%
1.50	0.1538	-0.1849	0.1922	4.09%
1.75	0.1674	-0.1431	0.2048	4.18%
2.00	0.1775	-0.1083	0.2062	4.26%
2.25	0.1846	-0.0793	0.1999	4.34%
2.50	0.1895	-0.0550	0.1882	4.41%
2.75	0.1926	-0.0346	0.1728	4.48%
3.00	0.1942	-0.0174	0.1549	4.54%
3.25	0.1947	-0.0030	0.1356	4.59%
3.50	0.1942	0.0090	0.1155	4.65%
3.75	0.1931	0.0191	0.0950	4.70%
4.00	0.1914	0.0275	0.0746	4.75%
4.25	0.1892	0.0344	0.0546	4.79%
4.50	0.1867	0.0401	0.0350	4.83%
4.75	0.1839	0.0448	0.0161	4.87%
5.00	0.1810	0.0485	-0.0020	4.91%
5.25	0.1779	0.0515	-0.0194	4.94%
5.50	0.1747	0.0538	-0.0359	4.97%
5.75	0.1715	0.0555	-0.0517	5.00%
6.00	0.1682	0.0567	-0.0666	5.03%
6.25	0.1649	0.0575	-0.0807	5.06%
6.50	0.1616	0.0580	-0.0940	5.08%
6.75	0.1584	0.0581	-0.1066	5.10%
7.00	0.1552	0.0580	-0.1184	5.13%
7.25	0.1520	0.0576	-0.1296	5.15%
7.50	0.1489	0.0571	-0.1401	5.17%
7.75	0.1459	0.0564	-0.1500	5.19%
8.00	0.1429	0.0556	-0.1593	5.21%
8.25	0.1400	0.0547	-0.1680	5.22%
8.50	0.1372	0.0536	-0.1763	5.24%
8.75	0.1344	0.0525	-0.1840	5.25%
9.00	0.1317	0.0514	-0.1913	5.27%
9.25	0.1290	0.0502	-0.1982	5.28%
9.50	0.1265	0.0490	-0.2047	5.30%
9.75	0.1240	0.0477	-0.2108	5.31%
10.00	0.1215	0.0464	-0.2166	5.32%

Table 5: Calculation of bond exposures to principal components on 27 May 2002

27 May 2002	Coupon	Maturity	Coupon dates	Time to coupon	Interpolated yields	Discount factor	Cash flows	Discounted cash flows	$u_1(t_i)$	$u_2(t_i)$	$u_3(t_i)$	$\sum_{i=1}^{N} P_{i(0)} C_i u_k(t_i) t_i$			Exposure U_1	Exposure U_2	Exposure U_3
Bond to be hedged	5.25	01/04/08	01/04/03	0.61	3.71%	0.9777	5.25	5.13	0.06	-0.43	-0.07	0.17	-1.34	-0.23			
	5.25	01/04/08	01/04/04	1.61	4.13%	0.9358	5.25	4.91	0.16	-0.17	0.20	1.26	-1.32	1.56			
	5.25	01/04/08	01/04/05	2.61	4.44%	0.8906	5.25	4.68	0.19	-0.05	0.18	2.33	-0.56	2.21			
	5.25	01/04/08	01/04/06	3.61	4.67%	0.8448	5.25	4.44	0.19	0.01	0.11	3.10	0.22	1.71			
	5.25	01/04/08	01/04/07	4.61	4.85%	0.7997	5.25	4.20	0.19	0.04	0.03	3.59	0.82	0.52			
	5.25	01/04/08	01/04/08	5.61	4.98%	0.7560	105.25	79.57	0.17	0.05	-0.04	77.36	24.35	-19.14			
								102.93				87.82	22.17	-13.37	5.40	1.36	-0.82
CTD of Schatz	4.25	03/12/04	03/12/03	0.79	3.79%	0.9704	4.25	4.12	0.08	-0.36	0.03	0.28	-1.18	0.10			
	4.25	03/12/04	03/12/04	1.79	4.19%	0.9275	104.25	96.70	0.17	-0.14	0.21	29.33	-23.86	35.58			
								100.82				29.61	-25.04	35.68	1.86	-1.57	2.24
CTD of Bobl	6	01/04/07	01/04/03	0.61	3.84%	0.9769	6	5.86	0.06	-0.43	-0.07	0.20	-1.53	-0.27			
	6	01/04/07	01/04/04	1.61	4.13%	0.9358	6	5.61	0.16	-0.17	0.20	1.44	-1.50	1.79			
	6	01/04/07	01/04/05	2.61	4.44%	0.8906	6	5.34	0.19	-0.05	0.18	2.66	-0.64	2.53			
	6	01/04/07	01/04/06	3.61	4.67%	0.8448	6	5.07	0.19	0.01	0.11	3.55	0.25	1.95			
	6	01/04/07	01/04/07	4.61	4.85%	0.7997	106	84.77	0.19	0.04	0.03	72.49	16.49	10.44			
								106.66				80.34	13.06	16.44	4.76	0.77	0.97
CTD of Bund	5.25	01/04/11	01/04/03	0.61	3.71%	0.9777	5.25	5.13	0.06	-0.43	-0.07	0.17	-1.34	-0.23			
	5.25	01/04/11	01/04/04	1.61	4.13%	0.9358	5.25	4.91	0.16	-0.17	0.20	1.26	-1.32	1.56			
	5.25	01/04/11	01/04/05	2.61	4.44%	0.8906	5.25	4.68	0.19	-0.05	0.18	2.33	-0.56	2.21			
	5.25	01/04/11	01/04/06	3.61	4.67%	0.8448	5.25	4.44	0.19	0.01	0.11	3.10	0.22	1.71			
	5.25	01/04/11	01/04/07	4.61	4.85%	0.7997	5.25	4.20	0.19	0.04	0.03	3.59	0.82	0.52			
	5.25	01/04/11	01/04/08	5.61	4.98%	0.7560	5.25	3.97	0.17	0.05	-0.04	3.86	1.21	-0.95			
	5.25	01/04/11	01/04/09	6.61	5.09%	0.7141	5.25	3.75	0.16	0.06	-0.10	3.97	1.44	-2.47			
	5.25	01/04/11	01/04/10	7.61	5.18%	0.6743	5.25	3.54	0.15	0.06	-0.14	3.98	1.53	-3.89			
	5.25	01/04/11	01/04/11	8.61	5.25%	0.6364	105.25	66.99	0.14	0.05	-0.18	78.44	30.67	-103.67			
								101.60				100.70	32.67	-105.22	6.27	2.03	-6.55

4.5. Combination hedge

The objective is to hedge an intermediate bond, whose maturity lies between the maturities of the standard hedging instruments. In hedging, the value of the position should not change in spite of interest rate movements. Using the basis point values and dollar durations:

$$-N_1 \cdot BPV_1 \cdot dY_1 - N_A \cdot BPV_A \cdot dY_A - N_B \cdot BPV_B \cdot dY_B = 0 \tag{36}$$

where N_I is the nominal value of the bond to be hedged and N_A, N_B are nominal values of the standard hedging instruments. dY_I, dY_A, and dY_B are the corresponding yield changes. In this yield curve model also changes in the slope of the curve are possible. Therefore also the following equation must hold so that the combined positions would not be affected by changes in the yield curve:

$$N_1 \cdot BPV_1 \cdot [dY_A + ds(Y_1 - Y_A)] + N_A \cdot BPV_A \cdot dY_A + N_B \cdot BPV_B \cdot [dY_A + ds(Y_B - Y_A)] = 0 \tag{37}$$

where ds is a small change in the slope of the yield curve. Hedge ratios, i.e. the amount of futures contracts that should be sold, can be solved from the previous equations:

$$N_A = -\frac{N_1 \cdot BPV_1 \cdot (Y_B - Y_1)}{BPV_A \cdot (Y_B - Y_A)} \tag{38}$$

$$N_B = -\frac{N_1 \cdot BPV_1 \cdot (Y_1 - Y_A)}{BPV_A \cdot (Y_B - Y_A)} \tag{39}$$

The difference between these hedge ratios and the hedge ratios given by the duration model is that the yield spreads come into play in these equations.

There will be problems with these hedge ratios, if the yield Y_I is lower (higher) than Y_A (Y_B) although the modified duration of bond to be hedged is higher (lower) than that of the hedge instrument. The solution used in the study is to restrict the yield level of the bond to be hedged (Y_I) so that it cannot fall below (rise above) the yield of the hedge instrument

4.6. Risk-point method

In the risk-point method, the basic procedure is to divide a coupon bond into the discounted cash flows it generates i.e. all the coupons and the principal are valued separately. To value the cash flows, discount rates are needed. The correct discount rates are zero coupon yields. The method is implemented according to the procedures used by Dattatreya and Fabozzi (1995) and therefore bootstrapping is used to calculate the zero yields, not the estimated Nelson-Siegel zero yields that are used with the principal components analysis. The reason for this choice is that in the risk-point method the changes in the yields that are used are specifically changes in the hedging instrument yields. Therefore also the yield curve is bootstrapped using the market quotes for the same three CTD-bonds at any time. The major differences in these zero coupon yield estimation methods are that in Nelson-Siegel estimation the yield curve is constructed using 15-20 liquid bonds. Now only quotes for three hedging instruments are used and the bootstrapping is based solely on quotes on these three CTD-bonds. The bootstrapping is a powerful method in valuation of bonds and suits for this purpose quite well. Bootstrapping produces only three zero yield quotes for the whole maturity spectrum. The yield curve is approximated to be linear between the maturities of the quotes each day.

The hedge instruments are Schatz, Bobl and Bund futures as with all the methods. These futures do not have cash flows, but cash flows of the underlying cheapest to deliver bond can be used to obtain the characteristics of the hedge instruments. The problem is that the CTD-bond may change constantly, which can cause problems to the hedge. This can make the results worse, since it increases the amount of transactions suggested by the method. However, Dattatreya and Fabozzi (1995) state that although at first it seems that the method is quite dependent on the hedge instruments chosen, in reality the method is quite robust, and under most conditions handles arbitrary selection of hedge instruments well.

An essential part of the risk-point method is a model that values the assets relative to the prices of the hedge instruments chosen. The yield and therefore the price of one of the hedge instruments are changed by a small amount, in the model by 1 basis point, and then the zero yields are recalculated. The value of the portfolio to be hedged can now be calculated again using the new zero yields as discount rates. The change in the value of the bond to be hedged relative to the change in the yield of the hedge gives the risk point relative to that hedge instrument. The change in the value is the risk-point of the portfolio. The method gives the amount of the hedge instrument needed for hedging by equating the risk-point of the hedge to the risk-point of the portfolio. The procedure of changing the yields of the hedge instruments, calculating the zero rates again and revaluing the cash flows of the bond to be hedged is repeated for all three hedge instruments chosen.

For example the risk-point against Bobl futures contract can be obtained by increasing the yield of the futures contract, i.e. the underlying CTD-bond, by one basis point, which causes the price to decline e.g. from 103.71 to 103.67, that is by 0.04. This number is the change in dollars for every €100 par holding of the bond when the hedging instrument yield changes by 1 basis point. Therefore it is the risk-point of the bond that is being hedged relative to the five-year bond underlying the Bobl futures contract. The table 6 presents an example of the estimated zero yield curve and table 7 an example of braking the bonds into cash flows and valuing them separately.

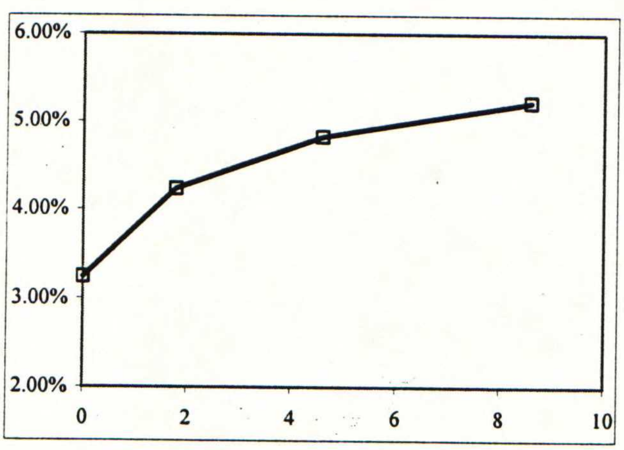
The collection of risk-points represents the hedge portfolio. The number of futures contracts needed for the hedge is calculated as follows: After the risk-point of the bond to be hedged is known, the risk-point of the hedge instrument relative to itself is calculated. This figure is very close to the BPV of the bond, but differs in that the calculation does not begin with a parallel yield curve. Then the corresponding risk-point of the bond to be hedged is divided by the risk-point of the hedge instrument relative to itself. This calculation gives the amount of CTD-bond that should be sold short as a part of the hedge in order to hedge the position. Since we are using futures contracts, this factor is multiplied by the conversion factor of the CTD-bond to get the amount of futures contract that should be sold short.

The quotes used are the market closing quotes for current CTD-bonds for Schatz, Bobl and Bund futures contracts as provided by Bloomberg. In addition, a yield quote with one-day horizon is needed to estimate the yield curve between the maturity of zero and the Schatz maturity. EONIA overnight rate could be used for this purpose, but it tends to be quite volatile from time to time. Usually it however follows very closely the European Central Bank (ECB) minimum bid rate, which is chosen to approximate a yield with one-day horizon.

Table 6: Example of market quotes and the estimated zero yield curve as used in risk-point method

One-day yield is approximated with the daily ECB minimum bid rate quote. Estimated zero coupon yields are estimated using bootstrapping-method. Maturity is expressed in a form mm/dd/yy and the time to maturity of a bond is in years.

	Market quotes		
May 31st 2002	Schatz - 2Y sector	Bobl - 5Y sector	Bund - 10Y sector
CTD-bond maturity	03/12/04	01/04/07	01/04/11
CTD-bond yield	4.24%	4.80%	5.17%
CTD-bond coupon	4.25	6	5.25
CTD-bond time to maturity	1.79	4.61	8.61



Estimated zero coupon yields	
Maturity	Zero yield
0	3.250%
1.79	4.246%
4.61	4.844%
8.61	5.244%

Table 7: Example of calculations needed to obtain the risk-point hedge ratios

The bond to be hedged matures at 01/04/2008, has a coupon of 5.25%. Calculations are from 27 May, 2002

Bond to be hedged, original yields						Schatz CTD yield +1 basis point			Bobl CTD yield +1 basis point			Bund CTD yield +1 basis point		
Coupon dates	Time to coupon	Zero yields	Discount factor	Cash flows	Discounted cash flows	New zero yields	Discount factor	Discounted cash flows	New zero yields	Discount factor	Discounted cash flows	New zero yields	Discount factor	Discounted cash flows
01/04/03	0.61	3.59%	0.979	5.25	5.139	3.59%	0.979	5.139	3.59%	0.979	5.139	3.59%	0.979	5.139
01/04/04	1.61	4.14%	0.937	5.25	4.918	4.15%	0.937	4.918	4.14%	0.937	4.918	4.14%	0.937	4.918
01/04/05	2.61	4.42%	0.893	5.25	4.689	4.43%	0.893	4.689	4.42%	0.893	4.689	4.42%	0.893	4.689
01/04/06	3.61	4.63%	0.849	5.25	4.458	4.64%	0.849	4.458	4.64%	0.849	4.457	4.63%	0.849	4.458
01/04/07	4.61	4.84%	0.804	5.25	4.221	4.84%	0.804	4.221	4.86%	0.804	4.219	4.84%	0.804	4.221
01/04/08	5.61	4.94%	0.763	105.25	80.283	4.94%	0.763	80.283	4.95%	0.762	80.249	4.95%	0.763	80.271
					103.7088			103.7066			103.6711			103.6964

CTD bond valuation for Schatz futures						Schatz CTD yield +1 basis point		
Coupon dates	Time to coupon	Zero yields	Discount factor	Cash flows	Discounted cash flows	New zero yields	Discount factor	Discounted cash flows
03/12/03	0.79	3.69%	0.972	4.25	4.130	3.69%	0.972	4.130
03/12/04	1.79	4.25%	0.928	104.25	96.755	4.26%	0.928	96.738
					100.884			100.867

CTD bond valuation for Bobl futures						Bobl CTD yield +1 basis point		
Coupon dates	Time to coupon	Zero yields	Discount factor	Cash flows	Discounted cash flows	New zero yields	Discount factor	Discounted cash flows
01/04/03	0.61	3.59%	0.979	6	5.873	3.59%	0.979	5.873
01/04/04	1.61	4.14%	0.937	6	5.621	4.14%	0.937	5.621
01/04/05	2.61	4.42%	0.893	6	5.359	4.42%	0.893	5.359
01/04/06	3.61	4.63%	0.849	6	5.095	4.64%	0.849	5.094
01/04/07	4.61	4.84%	0.804	106	85.226	4.86%	0.804	85.186
					107.174			107.132

CTD bond valuation for Bund futures						Bund CTD yield +1 basis point		
Coupon dates	Time to coupon	Zero yields	Discount factor	Cash flows	Discounted cash flows	New zero yields	Discount factor	Discounted cash flows
01/04/03	0.61	3.59%	0.979	5.25	5.139	3.59%	0.979	5.139
01/04/04	1.61	4.14%	0.937	5.25	4.918	4.14%	0.937	4.918
01/04/05	2.61	4.42%	0.893	5.25	4.689	4.42%	0.893	4.689
01/04/06	3.61	4.63%	0.849	5.25	4.458	4.63%	0.849	4.458
01/04/07	4.61	4.84%	0.804	5.25	4.221	4.84%	0.804	4.221
01/04/08	5.61	4.94%	0.763	5.25	4.005	4.95%	0.763	4.004
01/04/09	6.61	5.04%	0.722	5.25	3.791	5.05%	0.722	3.790
01/04/10	7.61	5.14%	0.683	5.25	3.583	5.15%	0.682	3.581
01/04/11	8.61	5.24%	0.644	105.25	67.768	5.26%	0.643	67.704
					102.573			102.505

4.7. Constructing the hedged portfolio

First single bonds are hedged using all the five hedge ratio estimation methods. After that, portfolios are constructed of these single bonds. In order to assure that every bond has approximately equal contribution to the whole portfolio's return, the bonds are weighted using factor [1 / *modified duration*] of a bond as a weight. This procedure is adapted from Litterman and Scheinkman (1991). The portfolio weights are adjusted at the beginning of each year.

When the hedge ratios are calculated for each hedging model, the portfolio of long bonds and short futures contracts is constructed. The return on this hedged portfolio is calculated as presented in Hilliard (1984):

$$r_p = \frac{P_1 - P_0 + F_1 - F_0}{P_0} = r_s + \frac{F_0}{P_0} \cdot r_f \tag{40}$$

where r_p is the return on the hedged portfolio, P_1 is the long bond portfolio value today, P_0 is the long bond portfolio value yesterday, F_1 is the futures price today, F_0 is the futures price yesterday. Transaction costs are ignored and it is assumed that sufficient collateral is available to establish the futures position with no additional capital requirements. The total yield is thus the holding period yield on the spot position (r_s) plus the ratio of the futures price to spot position times the price yield on the futures contract ($r_f = (F_1 - F_0) / F_0$). In general, for a portfolio of *n* cash and *m* futures positions, the portfolio yield equation can be written as

$$r_p = \sum_{i=1}^{n} \alpha_i \cdot r_{si} + \sum_{j=1}^{m} \frac{F_{0j}}{P_{0j}} \cdot r_{fj} \tag{41}$$

where α_i are spot portfolio weights.

The single bonds and the constructed portfolios are hedged using

- Only one hedging instrument at a time while duration or regression-based method are used
- Two or three hedge instruments at a time while PCA, combination hedge or risk-point method are used

The futures contracts expire on the 10th of March, June, September and December. In this study, the portfolios are hedged using the futures contract closest to the expiration, since this contract tends to have the highest liquidity. The roll to the next contract is done on a last Monday of February, May, August and November. This is done in order to avoid the special factors that may affect to the pricing of the futures contracts when they approach the maturity, for example so called short squeeze.

The fact that futures contracts are used as the hedge instruments, causes three sources of inefficiency to the models. First, the hedge ratio estimation models are executed as if the cheapest to deliver bond would be sold short, instead of the futures contract. In addition to conversion factors, no special adjustments are made to take into account the fact that we are using futures contracts (except with the duration hedge). Second, since the hedge position will be held only for a week at a time, the behavior of the basis is uncertain. The basis may widen or narrow, which causes the hedged position to gain or lose. Third, basis converges towards zero in delivery. Since the hedges are rolled forward each week, the position suffers from the basis convergence, although the positions roll to the next contract a couple of weeks before the expiry. These factors will most likely weaken the performance of the hedges in the empirical study.

4.8. Descriptive data

In this chapter, Nelson-Siegel estimated zero yield curve data are used for the data description purposes. The chapter discusses the changes that have been observed on the German yield curve during the sample period 01/01/1999 – 05/31/2002. In figure 6, the German zero yields are depicted. As can be seen, the estimation period includes two periods when yields are generally rising, and therefore hedging can be hypothesized to increase the hedged portfolios returns during these periods, and one longer period when interest rates are generally declining, when constant hedging decreases the returns on the portfolios. However, the interest rate differentials between two-year, five-year, and ten-year yields are far from constant, which support the claim that hedging against yield curve shape changes may be justified in order to reduce the interest rate risk as much as possible.

Figure 6: German zero yields during the estimation period

The lines depict the German two-year, five-year and ten-year zero-coupon yields during the estimation period in this study, 01/01/99 - 05/31/02

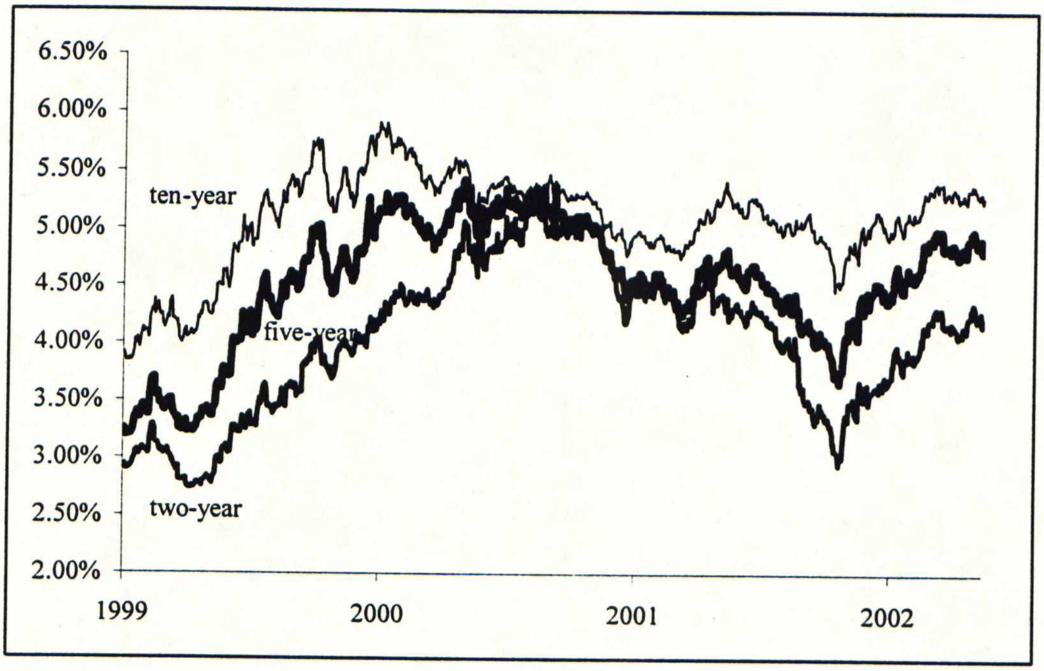


Table 8: Descriptive yield data from the sample period 01/01/1999 - 05/31/2002

The data consists of Nelson-Siegel estimated German zero yield curve data from the sample period. Yield curve steepness means yield spreads between different maturities. Column labeled 2-5Y describes yield spreads between two-year yield and five-year yield. The columns 5-10Y and 2-10Y are constructed similarly. Curvature means yield spread between five-year zero yield and the yield that lies on a straight line drawn between two-year yield and ten-year yield. The column labeled 2-5-10Y describes the curvature data during the sample period.

	Yield level									
	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
Min	2.615%	2.742%	2.883%	3.051%	3.199%	3.331%	3.465%	3.597%	3.724%	3.844%
Max	5.217%	5.316%	5.347%	5.363%	5.429%	5.491%	5.632%	5.745%	5.836%	5.912%
Average	3.826%	4.007%	4.188%	4.359%	4.513%	4.650%	4.772%	4.879%	4.973%	5.056%
Variance	0.005%	0.004%	0.004%	0.003%	0.003%	0.003%	0.002%	0.002%	0.002%	0.002%

	Yield curve steepness			Curvature
	2-5Y	5-10Y	2-10Y	2-5-10Y
Min	-0.04%	0.02%	0.07%	-0.145%
Max	1.12%	0.96%	1.90%	0.502%
Average	0.51%	0.54%	1.05%	0.112%
Variance	0.0008%	0.0005%	0.0020%	0.000%

Table 8 presents descriptive data on zero yields behavior during the estimation period. In addition to yield level changes, also yield curve shape changes are considered in a form of yield curve steepness and curvature changes. Yield curve steepness is measured using yield spreads between maturities of two and five years, five and ten years, and two and ten years. The changes in yield curve steepness have been quite significant. For example, at the other extreme, yield curve has been inverted between two and five years and at the other extreme the yield spread between these maturities has been 1.12%. Yield curve curvature changes are measured as the yield spread between the five-year yield and the yield that lies under the five-year yield on a straight line drawn from two-year to ten-year yield. The variation in curvature seems also quite significant and it therefore seems possible that these factors describing the yield curve shape changes may have a significant effect on the returns on the hedged bond portfolios.

The problem with the duration hedge is that it is not generally capable of hedging against yield curve shape changes, which is a major reason why other hedging methods are considered to possibly result in a better hedge. To get a better picture of the yield curve shape variation during the estimation period, the table 9 presents the largest weekly yield changes.

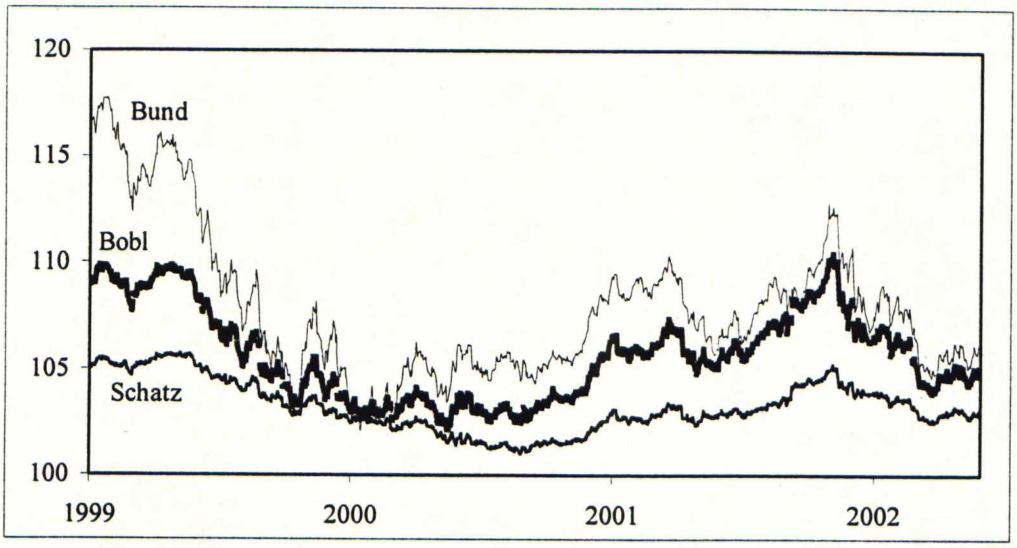
Table 9: Largest changes in yield curve shape

The changes are from Nelson-Siegel estimated zero yield curve. In column labeled 2-5Y is presented 15 largest weekly changes of yield spreads between two-year yield and five-year yield during the period 01/01/1999 - 05/31/2002. The columns 5-10Y and 2-10Y are constructed similarly. Curvature means yield spread between five-year zero yield and the yield that lies on a straight line drawn between 2-year yield and 10-year yield. The column labeled 2-5-10Y describes the 15 largest weekly changes in the curvature observed during the sample period.

Steepness						**Curvature**	
2-5Y		5-10Y		2-10Y		2-5-10Y	
01/03/2000	0.330%	09/18/2000	0.252%	09/18/2000	0.430%	01/03/2000	0.268%
01/10/2000	0.306%	01/10/2000	0.184%	05/14/2001	0.329%	01/10/2000	0.260%
12/27/2000	0.207%	09/17/2001	0.178%	05/28/2001	0.294%	12/27/2000	0.117%
05/28/2001	0.183%	01/03/2000	0.164%	09/17/2001	0.278%	10/18/1999	0.107%
05/14/2001	0.182%	05/14/2001	0.147%	03/29/1999	0.253%	01/08/2001	0.097%
09/18/2000	0.178%	03/08/1999	0.142%	05/08/2000	0.246%	02/07/2000	0.093%
10/18/1999	0.172%	03/29/1999	0.134%	12/27/2000	0.241%	04/25/2000	0.089%
09/24/2001	0.168%	07/31/2000	0.128%	08/07/2000	0.228%	09/24/2001	0.088%
02/07/2000	0.161%	05/22/2000	0.124%	08/14/2000	0.219%	12/10/2001	0.088%
11/01/1999	0.150%	05/08/2000	0.122%	09/24/2001	0.212%	07/24/2000	0.080%
12/06/1999	0.140%	01/31/2000	0.118%	05/22/2000	0.187%	11/01/1999	0.080%
07/24/2000	0.132%	05/28/2001	0.111%	11/01/1999	0.186%	05/15/2000	0.077%
01/08/2001	0.130%	01/28/2002	0.108%	03/20/2000	0.183%	12/06/1999	0.076%
05/08/2000	0.123%	08/07/2000	0.107%	02/07/2000	0.180%	05/29/2000	0.075%
03/20/2000	0.123%	08/14/2000	0.105%	04/17/2001	0.179%	05/28/2001	0.073%

Figure 7: Schatz, Bobl and Bund futures prices

The figure presents the development of the futures prices during 01/01/1999 - 05/31/2002.



The futures prices follow very closely the price of the underlying cheapest to deliver bond. In figure 7 the Schatz, Bobl, and Bund futures prices are depicted.

4.9. Principal component analysis' description of the German yield curve

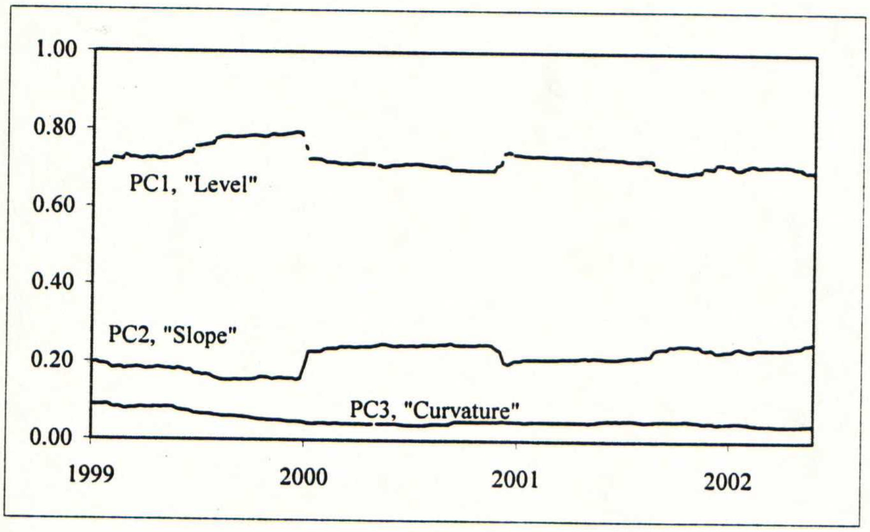
Principal component analysis is performed to gain further insights of the factors that drive yield curve changes. The idea behind the method is that it is most reasonable to hedge the portfolio against changes of those factors that affect most to the yield curve changes. In previous studies by Barber and Copper (1996) and Baygün, Showers, Charpelis (2001) the three first principal components are able to explain 97-99% of the variation in the yield curve. Both Barber and Copper (1996) and Baygün, Showers, Charpelis (2001) studied the US yield curve.

The results of the principal components analysis are mostly very well in line with those obtained from the previous studies. Both Barber and Copper (1996) and Baygün, Showers, Charpelis (2001) provide only one set of principal components. Since in this study the performance of hedge ratio estimation methods are compared, the principal components are estimated weekly with rolling three-year estimation period in order to avoid in-sample estimation of the hedge ratios. Therefore, compared to the previous studies, this study can provide some guidance on the stability of the principal components over time. From the results it seems that the explanatory power of the three first principal components remains quite stable over the estimation period, regardless of the fact that each estimation is performed with slightly different data set. Over the 176 weekly observations, the first three principal components are able to explain a minimum of 99.2% and a maximum of 99.9% of the variation in the yield curve. On average, the three first principal components were able to explain 99.6% of the variation, which means that remaining 37 principal components are able to explain on average the additional 0.4%. It seems, that three principal components describe adequately the variation in the German yield curve.

Figure 8: Proportion of total variance in weekly yield changes explained by the three first principal components

The table describes how large proportion of total variance in weekly German zero yield curve changes is explained by the first three principal components during the period 01/01/1999 - 05/31/2002. The chart describes the weekly variation in the explanatory power of the first three principal components.

	PC1	PC2	PC3	Sum
Average	72.65%	21.58%	5.33%	99.57%
Min	69.17%	15.38%	3.87%	99.21%
Max	79.42%	25.56%	9.05%	99.87%
St dev	2.62%	3.16%	1.32%	0.17%



In figure 8, also the explanatory power of the first components individually appears to be relatively stable. The only more noticeable changes can be observed in the beginning of 2000 and 2001. At the beginning of 2000 the factor loadings of PC2 changed signs. In the beginning and in the end of the estimation period the shape of the PC2 was upward sloping but it turned to downward sloping in between. However, the absolute values of the factor loadings remained stable in spite of this change of sign.

The variation of factor loadings can be observed from table 10 that presents factor loadings of ten selected maturities out of the estimated 40 maturities. The standard deviations of the values that first principal component's factor loadings get with different estimations seem quite small. More variation is in the shorter end. The "slope"-component signs for all factor loadings change on 01/03/2000 and then again on 12/18/2000, so the averages and standard deviations presented in the table do not give a thorough picture of the absolute values of the factor loadings.

Table 10: Factor loadings in principal components.

The factor loadings of ten different maturities are presented in the table for the first three principal components. The loadings are estimated from covariance matrices of German weekly zero yield changes. The estimation period were 01/01/1999 - 05/31/2002, which includes 177 weekly observations. PC2 changes the signs of the factor loadings for period 01/03/2000 - 12/18/2000, so the average values and standard deviations do not describe the factor's true behavior.

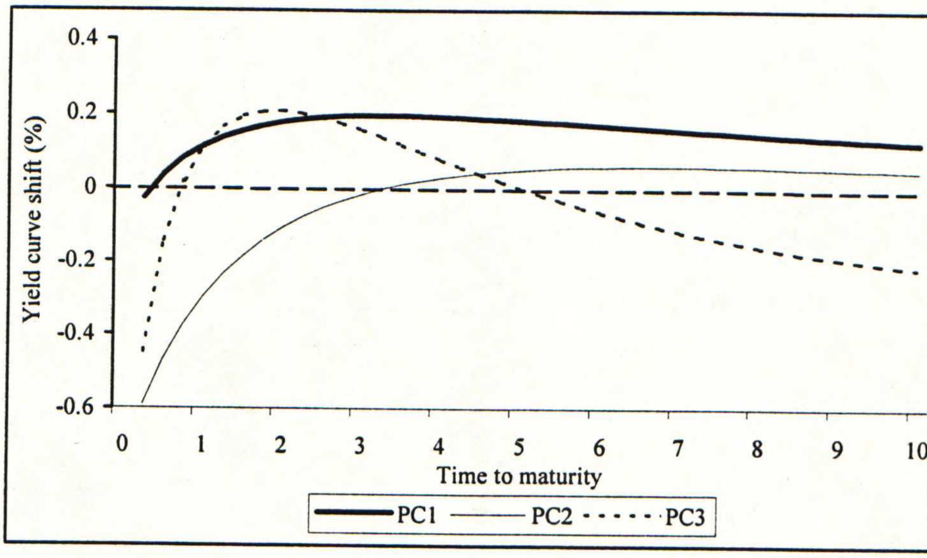
PC1	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
Min	0.049	0.134	0.170	0.180	0.176	0.166	0.155	0.143	0.132	0.122
Max	0.112	0.177	0.194	0.192	0.187	0.182	0.175	0.166	0.158	0.150
Average	0.083	0.152	0.179	0.185	0.183	0.176	0.168	0.158	0.149	0.141
St dev	0.020	0.014	0.007	0.003	0.003	0.004	0.006	0.007	0.008	0.008

PC2	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
Min	-0.303	-0.115	-0.029	-0.028	-0.035	-0.036	-0.035	-0.033	-0.030	-0.027
Max	0.299	0.106	0.025	0.031	0.049	0.057	0.058	0.056	0.052	0.048
Average	-0.098	-0.037	-0.006	0.011	0.019	0.024	0.026	0.026	0.025	0.025
St dev	0.281	0.095	0.019	0.017	0.029	0.033	0.032	0.029	0.026	0.022

PC3	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
Min	0.064	0.206	0.155	0.045	-0.039	-0.096	-0.133	-0.163	-0.206	-0.239
Max	0.183	0.263	0.202	0.124	0.039	-0.039	-0.102	-0.150	-0.169	-0.177
Average	0.106	0.224	0.182	0.102	0.020	-0.052	-0.110	-0.156	-0.192	-0.219
St dev	0.021	0.015	0.014	0.019	0.018	0.013	0.007	0.003	0.009	0.015

Figure 9: Example of the factor loadings, 27 May 2002

The lines depict the factor loadings of the principal components on 27 May 2002.



In figure 9 factor loadings of the principal components are depicted. The factor loadings describe the yield change that a unit change in the particular factor would cause. The principal component that explains most of the variation (PC1) is quite stable after the very short end of the yield curve. Therefore previous researchers have labeled it a "level" component. It explains most of the variation in the changes of the yield curve, for example 69.6% during the prior three-year estimation period before the 27th of May 2002. Table 10 indicates that the factor loadings tend to remain stable over time, and therefore the principal components could be able to provide solid performance for hedging purposes. The principal component that explains most variation after the "level" component, PC2, changes it sign from negative to positive approximately between maturities of three and four years. Therefore changes in this factor cause yields on the short end and on the long end of the yield curve to move in different directions. The fact has led previous researchers to label the factor as the "slope" of the curve. This component explained on average further 25.5% of the changes in the yield curve during the three-year estimation period. In this study, the PC2 changed it shape from upward sloping to downward sloping for the estimation period of 01/03/2000 - 12/04/2000. However, the absolute values of the factor loadings did not change much during this change. It should also be noted that this change does not have major implications to the hedging performance, since the hedge is based on portfolio's exposure to the factors relative to the hedge instruments exposure to the same factors. If the absolute values remain the same, also the hedge ratios will.

The third factor (PC3) is positive between maturities of one and five years and negative elsewhere. Thus a positive realization of that factor tends to cause the "humbedness" of the yield curve to increase. This notion has led previous researchers to name this factor as "curvature". It should be noted that e.g. a positive realization of the PC1 is not a pure level change on the yield curve, although it is quite close to it. The labels that are given to these factors, however, describe the sources of variation in which yield curve changes often are categorized in, and the use of them makes the discussion of the principal component method easier to understand.

The results obtained from the PCA are similar in nature compared to those received by previous studies. In my study, the most notable difference compared to the other studies is that the first component explains notably less of the variance, and the second component clearly more. In my study, the first component explaines on average 73%, the second component 22% and the third component 5%, whereas the figures are respectively 89%, 8% and 2% for Litterman and Scheinkman (1991) and 81%, 12% and 4% for Barber and Copper (1996).

5. Empirical results

The section is divided into three parts. First, the comparison of the performance between the hedging methods is presented for single bonds and for bond portfolios. Second, an analysis of the remaining variance in the portfolios is provided. Third, a short summary of the most important results is presented.

5.1. Differences in the performance of the hedging methods

At first, the hedging performance of the different hedging methods over the single government bonds is presented. The data set includes 15 German government bonds, which all are hedged separately with each of these models. More detailed results of the hedging performance on the single bonds are provided in Appendix A. This chapter presents average figures of the single hedges over the estimation period of three years and five months and over the 15 hedged bonds.

5.1.1. Hedging single bonds

The table 11 presents the hedging performance of different hedge ratio estimation methods on average for the 15 bonds that all are hedged separately. The hedged portfolios include a long position in a government bond and a short position in the futures contracts. The amount of futures contracts is that implied by the hedging models. The amount of short futures contracts is recalculated weekly, and necessary adjustments are made to the hedged portfolios. These adjustments reflect to the number of futures transactions that the models imply, and since the cost of the hedge is of major importance, the needed amount of transactions is also presented in the table. The return data in the table 11 includes also coupons, which are reinvested in the bonds.

The previous studies on the hedging performance of duration have often concentrated on how well duration can explain returns in a bond portfolio and used regression analysis as a tool to explain duration's ability to explain returns in a bond portfolio. In this study, the remaining variance in the hedged portfolios includes also the variation caused by basis changes in the futures contracts, and the problems that are caused due to the fact that the cheapest to deliver bond's characteristics are used as if the hedging instrument would be the CTD-bond, not the futures contract. All the estimations are also performed using out-of-the sample data. Therefore, the results of this study can provide rather realistic view on the probable performance of an actual hedging decision, but since, as far as I know, studies with similar methodology have not been done so far, a comparison of the results to the previous studies can be more difficult.

In the original study of regression-based hedging, Ederington (1979) uses a simple method of measuring the reduction in the variance of the portfolio that is caused by the hedge. He defined the measure as [1 - (*variance of hedged portfolio / variance of unhedged portfolio*)]. In this study that measure is called the 'Ederington measure'. Ederington (1979) hedged 90-day T-bills with T-bills futures contracts for two-week periods, and was able to reduce the variance of the portfolio that was being hedged by only 27.2%. However, when Franckle (1980) did the same calculations again with the corrected data, the reduction in variance increased to 67.9%. In this study, the reduction in variance achieved by the regression-based hedge is clearly higher than what was achieved by Ederington (1979) or even Franckle (1980). In this study, the hedges succeed in removing on average 84% of the variance. However, the portfolios that use regression as the hedge ratio estimation method have 35% more remaining variance than the portfolios hedged with duration, and they use 33% less futures contracts. It seems probable that the hedge ratios in the regression hedges are too low, most likely due to problems in the estimation procedure. The maturity of the bond shortens all the time during the data period, which restricts the amount of data that can be reasonably included into each hedge ratio estimation.

The results indicate that the hedge constructed using PCA would be able to reduce the hedged portfolio's variance by 27% compared to the hedge constructed using duration. The results can be compared to the ones obtained earlier by Litterman and Scheinkman (1991), who also apply principal components analysis on hedging, but who estimate the components using historical excess returns and apparently hedge with in-sample data. Litterman and Scheinkman (1991) use a bond portfolio as a hedging instrument whereas in this study three futures contracts with differing maturities are used. Considering the differences in the studies

Table 11: Hedging performance of different methods on single bonds - average figures over the 15 bonds

The table describes the average hedging performance of five hedging methods, when 15 single German government bonds were hedged using bond futures contracts. The estimation period was 1/1/1999 - 5/31/2002. The figures are averages over the standard deviations and returns from the 15 single bonds. In each case, portfolio consist of a single bond and the corresponding amount of short futures contracts as suggested by the hedging methods. The appropriate amount of futures contracts is recalculated weekly and corresponding changes are made. The return is holding period return for the portfolio excluding transaction cost of futures trades. Coupons are reinvested in the bonds. Standard deviations and returns are annualized. Number of sold futures is average over all bond hedges and corresponds a nominal value of €100 000 of bonds. One futures transaction in the table includes both opening and closing the trade. The Ederington measure of hedging efficiency is calculated as follows: [1- (variance of hedged portfolio / variance of unhedged portfolio)].

*In 1999, regression hedge ratios are calculated for period 04/01/1999-12/31/1999 only.

			Ederington measure			Number of futures transactions			
	Std dev.	Return	Min	Average	Max	Schatz	Bobl	Bund	Sum
Duration	1.25%	4.22%	0.792	0.872	0.959	2.16	12.55	2.75	17.46
Regression*	1.45%	4.14%	0.782	0.836	0.898	1.47	8.80	1.38	11.65
PCA	1.07%	4.13%	0.831	0.904	0.973	3.99	10.34	4.97	19.31
Risk point	1.16%	4.16%	0.826	0.890	0.959	4.78	9.46	3.88	18.13
Combination hedge	1.16%	4.11%	0.666	0.883	0.950	3.37	11.92	5.58	20.86
Unhedged	3.67%	3.40%							

and the fact that the results in this study also include the additional deviations caused by the basis changes in the futures position, the results are surprisingly similar. With their approach, the hedge based on principal components was able to reduce the residual variance by an average of 28% compared to the duration hedge.

There does not seem to be any previous empirical evidence on the hedging performance of the risk-point method and the combination hedge. According to the results, the both methods are able to reduce the variance of the hedged portfolio on average by approximately 13% compared to that remaining in the duration-hedged portfolio. In addition, as Litterman and Scheinkman (1991) point out, the reduction in the variance obtained with the more advanced models is just an average reduction. In particular instances, such as in major yield curve shape changes, the improvement can be much greater.

In line with the hypothesis, the costs of the hedging seem to increase, when the shift is made from the duration-based hedge to the more advanced methods. According to the results, using PCA increases hedging costs approximately by 10%, using risk-point method by 4%, and using combination hedge by 20%.

Table 12: Differences in the hedging performance between duration and four other hedge ratio estimation methods, when 15 single bonds are hedged

The table describes how often better hedging performance was observed, compared to duration, using other hedge ratio estimation methods. The data consists of weekly observations of 15 German government bonds that were hedged separately in the study using bond futures contracts as hedge instruments. The estimation period was 1/1/1999 - 5/31/2002. The efficiency of the hedge is measured using the remaining standard deviation in the hedged portfolios.

*In 1999, regression hedge ratios are calculated for period 04/01/1999-12/31/1999 only.

Hedging method	With how many bonds the method was more efficient than duration	The percentage of times the method was more efficient than duration
Regression*	1	6.7%
PCA	15	100.0%
Risk point	13	86.7%
Combination hedge	9	60.0%

Table 12 presents some evidence on the stability of the superior or inferior hedging performance that the different hedging methods can provide. There are 15 bonds in the data set, and the bonds hedged with PCA had lower remaining variance in each of the 15 cases, compared to duration. The bonds hedged using the risk-point method had lower remaining variance in 13 cases, but the bonds hedged with combination hedge only with 9 cases out of 15. Combination hedge seems to produce most unstable hedging performance, while hedging single bonds. Regression-based hedges were inferior to the duration hedge in 14 cases out of 15. Appendix A provides more detailed information on the hedges of the 15 single bonds.

5.1.2. Hedging bond portfolios

In addition to the hedges of single bonds, I present results of the hedging performance on long bond portfolios that consist of all of the 15 bonds. The results are presented in table 13.

The results for the 15-bond portfolios are somewhat different compared to the results of the single bond hedges. It can be expected that the hedging effectiveness of duration and regression-based methods will decrease, since in these methods the portfolio duration is used as the risk measure, and practically during the whole estimation period all the bonds are hedged using the five-year Bobl futures contract. With PCA, risk-point method and combination hedge two or three hedging instruments are used in all the hedges. The results in table 13 are also broken down into returns and standard deviations of each year.

In general, the potential benefits of hedging become evident when the return figures for the unhedged portfolio are examined. In 1999, the return on the unhedged portfolio was negative i.e. the capital invested in bonds lost value in spite of the coupon flows, which is caused by the interest rate level rise during the year. Also on average over the whole estimation period all the hedged portfolios yielded higher return than the unhedged one, with considerably lower standard deviations.

Table 13: Hedging performance of different methods on bond portfolios

The portfolio consists of 15 German government bonds and the corresponding amount of short futures contracts as suggested by the hedging methods. The appropriate amount of futures contracts is recalculated weekly and corresponding changes are made to the short futures position. The hedges are constructed using the risk profile of the whole portfolio. Duration and regression based hedging methods include only one hedge instrument at the time. The weights of the long bonds in the portfolio are rebalansed in the beginning of each year. The weights equal [1 / modified duration of the bond]. The return is holding period return for the portfolio. Coupons are reinvested in the bonds. Standard deviations and returns are annualized. One sold futures contract corresponds a bond portfolio with nominal value of €100 000. One futures transaction in the table includes both opening and closing the trade.

*In 1999, regression hedge ratios are calculated for period 04/01/1999-12/31/1999 only.

	Whole period		1-5/2002		2001		2000		1999		Number of futures transactions, Whole period			
Hedging method	Std dev.	Return	Std dev.	Return	Std dev.	Return	Std dev.	Return	Std dev.	Return	Schatz	Bobl	Bund	Sum
Duration	1.17%	4.24%	0.40%	3.13%	0.64%	4.96%	0.98%	4.40%	1.82%	3.59%	0.951	17.875	0.000	18.827
Regression*	1.15%	3.94%	0.56%	2.74%	0.72%	5.31%	0.62%	5.03%	2.04%	2.32%	0.000	11.945	0.000	11.945
PCA	0.74%	4.20%	0.28%	3.18%	0.52%	4.87%	0.64%	4.15%	1.26%	4.23%	3.921	9.994	3.469	17.383
Risk point	0.82%	4.22%	0.27%	3.20%	0.55%	4.75%	0.74%	4.17%	1.42%	4.30%	5.561	9.236	3.195	17.992
Combination hedge	0.77%	4.19%	0.32%	3.22%	0.56%	4.88%	0.62%	4.22%	1.15%	4.04%	4.159	10.282	4.372	18.813
Unhedged	3.38%	3.51%	1.89%	1.75%	2.77%	6.06%	2.82%	6.81%	4.63%	-1.75%				

The methods that use more than one hedging instrument at a time, namely PCA, risk-point method and combination hedge, provide significantly better hedging performance than duration or the regression-based hedge. By using PCA instead of duration, the variance of the hedged portfolio can be reduced on average by 60% compared to the duration hedge. The average reduction in remaining variance for the risk-point method and the combination hedge are respectively 50% and 57%. Somewhat surprisingly and against the initial hypothesis, the cost of hedging does not increase compared to duration, although more than one hedging instrument is used, and the hedging performance is clearly enhanced.

Interestingly, the method based on regression is able to provide better hedging results than the duration-based hedge. This seems important also in the light that the regression-based hedge would have caused 33% less transaction costs. This phenomenon is quite interesting, and it will be considered further in the next section, which provides some explanations for the differences in the behavior of the hedged portfolios.

5.2. Remaining deviation of returns in the hedged portfolios

In this study, the purpose is to create hedged portfolios that are as little affected by changes in the yield curve as possible. To find out how well the different hedging methods succeed in immunizing the hedged portfolios against changes in yield level and yield curve shape, some further analysis is carried out. The analysis is performed by regressing the returns of the hedged portfolios against variables, which describe yield curve changes.

Table 14: Explaining the remaining variance in the hedged portfolios

The dependent variables are remaining weekly returns on the hedged portfolios on the period 01/01/1999 - 05/31/2002. The portfolios include 15 German government bonds that remain the same during the estimation period, but the weights of them are rebalansed in the beginning of each year. The weights equal [1 / modified duration of the bond]. The independent variables are 2-year yield level change (denoted "2Y"), the yield curve steepness change measured as 10-year yield - 2-year yield (denoted "2-10Y steepness") and the change in the yield curve curvature measured as yield spread of 5-year yield and the yield on a straight line drawn from 2-year yield to 10-year yield. All the independent variables are based on changes on Nelson-Siegel estimated zero yield curves. The figures without brackets are the regression coefficients, and the figures inside the brackets are their t-values.

Hedging method	Intercept	2Y yield	2-10Y steepness	2-5-10Y curvature	R^2
Unhedged	0.0010	-4.18	-2.39	-1.03	0.92
	(9.9)	(-36.45)	(-15.94)	(-3.63)	
Duration	0.0007	0.56	0.09	1.03	0.29
	(7.18)	(4.68)	(0.6)	(3.47)	
Regression	0.0008	-1.25	-0.94	0.29	0.53
	(9.4)	(-12.46)	(-7.19)	(1.15)	
PCA	0.0008	0.43	0.52	-0.10	0.19
	(10.13)	(4.82)	(4.44)	(-0.45)	
Risk point	0.0008	0.57	0.47	0.13	0.25
	(9.22)	(5.88)	(3.68)	(0.54)	
Combination hedge	0.0008	0.35	0.57	-0.31	0.17
	(10.73)	(4.07)	(5.16)	(-1.49)	

The first explanatory variable is yield level change, which is described using weekly two-year zero-yield change. The second variable describes yield curve steepening, which is measured using the weekly yield change between ten-year zero-yield and two-year zero-yield. The third explanatory variable is a change in the curvature of the yield curve, which is described using the weekly change of the five-year zero-yield over the straight line drawn from two to ten-year yields. The two-year yield level is chosen as the proxy for the yield level change to prevent linear correlation between the explanatory variables. It could be argued that e.g. the five-year yield would describe yield level changes better than the two-year yield, but then it would not be possible to use both the yield curve steepness change between two and ten years and the yield curve curvature change (2-5-10 years) as other explanatory variables, since the first two variables would already explain also the curvature change. As all of these three factors are however usually considered as independent sources of variation in the return of the portfolio, all of them were included in the regression and the two-year yield was chosen to proxy the yield level change. Table 14 presents the results of the regression.

The regression coefficient of the 2Y yield –variable that corresponds the yield level shift is an approximation of the remaining duration in the hedged portfolio. The modified duration is defined as

$$\frac{\Delta P}{P} = -MD \cdot \Delta y \tag{42}$$

where $\frac{\Delta P}{P}$ describes the percentage price change of the bond, *-MD* is the modified duration and Δy describes the yield level change. Since in the regression the dependent variables are the changes in the remaining returns of the hedged portfolios and the first independent variable is the yield level shift, the regression coefficient of the yield level variable approximates the duration of the portfolio. If the portfolios were properly hedged, the regression coefficient of this variable should be zero.

The regression coefficient of the yield level shift variable for the unhedged portfolio is –4.2, which approximates the average duration of the portfolio during the estimation period. In the unhedged portfolio, all three explanatory variables are highly significant, and the R^2 is very high, 0.92.

Interestingly the regressions seem to imply overhedging with most of the hedging methods. A positive realization of the two-year yield variable means that the yield has risen during the week. All the hedging models, except the regression method, have positive regression coefficients for the two-year variable, which implies that when yields rise, the value of the portfolio increases. Since value of a bond decreases when interest rates rise, the behavior of the hedged portfolios tends to suggest that the portfolios are overhedged against yield level changes, i.e. too many futures are sold short. Another interesting factor is that the estimated coefficients are quite close to each other, about 0.5, for each hedging model.

This finding is broadly in line with Ilmanen (1992). Ilmanen (1992) studies duration's explanatory power in explaining bond returns using regression and he concludes that duration seems to exaggerate the riskiness of a bond portfolio.

The fact that too many futures contracts were sold short could possibly be one major reason explaining the quite high holding period returns on the hedged portfolios that were observed during the period of increasing yields in the estimation period. The reason for this behavior of the hedging methods is not evident from these results. However, it should be noted, that the R^2 's are quite low for duration, PCA, risk point and combination hedge methods, which implies that clear majority of the remaining returns are caused by other factors than those used as explanatory variables. It is possible that the explanatory variables used in these regressions work as proxies to the unknown variables affecting the remaining returns in the hedged portfolios.

Out of the different hedging methods, the regression-based method is the only one that implied significantly lower hedge ratios for the portfolios. This factor is also reflected in the results. The two-year yield coefficient for this hedging method is highly significant and negative implying underhedging against yield level changes. According to these results, it would seem that the optimal hedge ratio against yield level changes would lie somewhere between those higher levels suggested by duration, principal components analysis, risk-point method and combination hedge method, and those clearly lower hedge ratios suggested by the regression-based hedging method. The reason for the too low hedge ratios provided by the regression-based method could be related to problems in the hedge ratio estimation. The longer the estimation window used to get better estimates, the more the maturity of the bond that is being hedged shortens.

The regression results for PCA, the risk-point method, and combination hedge seem broadly similar. The goodness of fit (R^2) of the regressions of the hedged portfolios decrease clearly from the 0.92 for the unhedged model to approximately 0.20 with these more advanced hedging methods, which implies that yield level changes and yield curve steepness or curvature changes can explain clearly less of the variation in the hedged portfolios' returns.

Table 15: Correlation matrix of the explanatory variables

The matrix presents Pearson correlation coefficients of weekly changes on Nelson-Siegel estimated German zero yield curves on the estimation period of 01/01/1999-05/31/2002.

	2Y yield	2-10Y steepness	2-5-10Y curvature
2Y yield	1.00		
2-10Y steepness	-0.50	1.00	
2-5-10Y curvature	0.15	0.49	1.00

The regression coefficients of 2-10Y steepness factor and the 2-5-10Y curvature factor should probably be interpreted carefully. The results obtained seem counterintuitive. PCA, risk-point method and combination hedge are the models that are supposed to be able to hedge the portfolio also against nonparallel yield curve shifts. According to the regression coefficients obtained, the steepness factor is significant in all of these models but zero in duration-based hedged, implying that the duration-based hedge is able to remove the hedged portfolio's exposure to steepness change in the yield curve, but the models that are supposed to do that are unable to do that. In addition, the hedging performance comparison between duration and the more advanced methods suggest that these models hedging performance is superior to that of duration's.

It is possible that multicollinearity between the explanatory variables have affected to the results. The correlation between the explanatory variables is quite high, see table 15. In addition, it is possible that factors related to the selection of the independent variables have affected the results. E.g. in the duration method, the regression coefficient of 2-5-10Y curvature could be significant because Bobl-futures were used almost solely for hedging with this method. Therefore also bonds with maturities of significantly below or above five years are hedged with the five-year futures contract, and the hedging portfolio that is sold short is very dependent of the five-year yield. If curvature increases, it means that five-year yield increases relative to the two-year and ten-year yields. When five-year contracts are those that are being sold short for most of the estimation period, this factor causes the short five-year position to outperform the long bond portfolio and possibly explains the quite high correlation of the duration-hedged portfolio and the curvature changes.

5.3. Summary of results

The results of this study could be interesting also from the point of view of a fixed income portfolio manager. The performance of the portfolio is usually measured against a benchmark index, and often the portfolio manager takes a view of lengthening or shortening the duration of his own portfolio compared to the index. The simplified actions of the portfolio manager are as follows: If the portfolio manager thinks the yields will decrease, he will lengthen the duration, if he thinks the yields will increase he will shorten the duration against the benchmark. The comparison of the hedging effectiveness of duration compared to other portfolio risk measures suggest that if the portfolio manager is only adjusting the duration of the fixed income portfolio while taking a view on the future course on interest rates, he will be exposed to risks that could possibly be taken into account if for example PCA, risk-point method, or combination hedge would be used to quantify the differences in the characteristics between the portfolio and the benchmark index.

Table 16 provides a short summary of the most significant results of this study.

Table 16: Summary table on the hedging performance of the models and the implied hedging costs

The table describes a summary of the hedging performance and the implied costs of five hedging methods. The data consists of weekly observations of 15 German government bonds that were hedged separately and in portfolios using bond futures contracts as hedge instruments. The estimation period was 1/1/1999 - 5/31/2002. Decrease in average variance is the difference between the remaining variance of the portfolio hedged with duration and the the remaining variance of the portfolio hedged with another hedging method. Average hedging costs are measured as the sum of the futures transactions implied by the models. The columns in the table describe how much hedging costs would increase compared to duration, if other hedging methods were used.

	Increase in average variance compared to duration		Increase in average hedging costs compared to duration	
	Single bonds	Portfolios	Single bonds	Portfolios
Duration	0.0%	0.0%	0.0%	0.0%
Regression	34.6%	-2.9%	-33.3%	-36.5%
PCA	-27.1%	-59.7%	10.6%	-7.7%
Risk-point	-13.3%	-50.4%	3.8%	-4.5%
Combination hedge	-13.8%	-56.9%	19.5%	-0.1%

6. Conclusions

The purpose of this study was to compare statistically the performance of different hedge ratio estimation methods that could be used to obtain the best possible hedge on a long government bond portfolio using futures contracts as hedging instruments. Although there are many models presented in the literature that should be able to take account also nonparallel yield curve shifts, there was very little statistical research on the relative performance of the different models, and maybe more importantly, on the performance in relation to the duration hedge, which is still the most widely used hedging method

It was hypothesized that by using the more advanced hedging methods, namely principal component analysis, risk-point method and combination hedge method, with two or three hedge instruments with differing maturities, it would be possible to decrease the remaining standard deviation in the hedged portfolio compared to the duration hedge or a traditional regression-based hedge with just one hedging instrument at a time. It was also hypothesized that the increased hedging performance would come at the cost of increased amount of futures transactions.

Principal component analysis as a hedging method uses historical yield curve shifts as a guide for probable future course of interest rates. Risk-point method is based on the idea that hedging could be performed relative to the characteristics of the hedging instruments. Combination hedge generalizes duration by taking yield curve steepness changes into account.

The empirical analysis in the thesis consisted of four main parts. First, principal components analysis was used to estimate three factors that were able to describe most of the variance on the German yield curve. Second, 15 single German government bonds were hedged for a period of three years and five months using five different methods for estimating the appropriate hedge ratios. Third, similar analysis was implemented with a German government bond portfolio that consisted of all the 15 bonds. The performance of the hedging method was measured with the remaining standard deviation in the hedged portfolios. Also the implied hedging costs, measured as the number of suggested futures transactions, were taken into account. The final part of the empirical study was analyzing the behavior of the returns that were still remaining in the hedged portfolios.

The estimation of the principal components yielded results that were well in line with those received by previous studies, namely Litterman and Scheinkman (1991) and Barber and Copper (1996), which both were estimated with US data. The explanatory power of the principal components that described the yield curve movements seemed to be slightly higher with the German yield curve than those obtained earlier by Barber and Copper (1996) with American data set. The explanatory power of the first principal component was lower, and the second component higher compared to the previous studies.

The three first principal components were able to explain over 99% of the variation in the weekly yield changes for the whole estimation period. The previous studies had estimated only one set of principal components, in my study the components were estimated weekly in order to be able to objectively compare the performance of the method during the hedging period. This resulted into 176 estimations of principal components. The results seemed to indicate that the estimated principal components tend to remain quite stable regardless of the sample.

In this study futures contracts were used as the hedging instruments. Futures contracts were considered since they are highly liquid, their transaction costs are low, and they are widely used in practice. Due to the characteristics of these contracts, it seemed natural that the hedging performance could be slightly weaker than what could be achieved by using government bonds as the instruments. However, the results of the study indicate that the achieved hedging performance was quite high. While using the traditional regression-based method for estimating the hedge ratios, the average reduction in the variance of the long bond portfolio was 84% for single bonds, which compares very favorably with Ederington (1979), who was able to reach 27% reduction in variance while hedging T-bills with T-bill futures. Even after corrections in the Ederington model, Franckle (1980) was able to reach only 68% reduction in the variance.

While using duration, probably the most widely used method for estimating the hedge ratio, to hedge single bonds, the average reduction in the hedged portfolio variance was 87% when measured using the Ederington measure.

The hedging method based on principal component analysis was able to further reduce the variance of the duration hedged bond by additional 27%, which is almost equal amount compared to the results obtained by Litterman and Scheinkman (1991). This similarity seemed slightly surprising considering that Litterman and Scheinkman (1991) used bond portfolios as a hedge instrument avoiding the deviations caused by the basis changes that are present in my study, in addition to further simplifying assumptions needed to implement the methods. The evidence from this study tends to imply that it is possible to achieve high hedging performance by using futures contracts as the hedge instruments.

The risk point method and the combination hedge were able to reduce the remaining variance by approximately 13% compared to duration. It seemed that the hedging performance of the risk-point method and combination hedge were not previously been statistically tested. According to the results, the cost of the hedge seemed to increase by 5-15% when the more advanced methods were used, which was in line with the hypothesis.

The assumption of parallel yield shifts was seen as a major problem of duration as a hedging method. The duration of a bond portfolio is defined as a weighted average of the durations of the individual bonds in the portfolio, with the weights being proportional to the bond prices. The problem with the assumption of parallel yield shift becomes significant in the context of portfolios, which consist of bonds with differing maturities. Since the cash flows are much more dispersed than in the single bond case, it is more and more likely that the parallel yield shift assumption causes more problems in the hedging performance. This phenomenon can also be found in the results obtained. The remaining variance of the portfolios hedged using duration matching could be decreased by 50-60%, when principal component analysis, risk-point method or combination hedge were used. Interestingly, this enhanced hedging performance was achieved together with decreased hedging costs. The comparison of the more advanced methods did not show any major differences in the performance of the models, but the performance of PCA seemed constantly slightly better than the other two models. The differences were however not especially significant.

When the remaining returns of the already hedged portfolios were explained by weekly yield curve changes, the results seemed to point to overhedging, i.e. that the models would constantly suggest executing too many futures transactions. This finding was broadly in line with Ilmanen (1992), who finds that duration tends to overestimate the riskiness of a bond portfolio.

However, it should be noted that the data sample used in this study could have been longer to obtain further certainty for the empirical results. Another possible weakness in the study arises from the fact that the special features of the futures contracts as hedging instruments were at some stages ignored and the futures contracts were treated merely as their behaved exactly like the corresponding cheapest to deliver bonds. This simplification may have weakened the results obtained.

References

Anderson, N., Sleath, J., 2001. New estimates of the UK real and nominal yield curves. Working Paper. Bank of England.

Barber, J.R., Copper, M.L., 1996. Immunization using principal component analysis. Journal of Portfolio Management, Summer.

Baygün, B., Showers, J., Cherpelis, G., 2000. Principles of principal components. SalomonSmithBarney.

Bierwag, G.O., 1977. Immunization, duration and the term structure of interest rates. Journal of Financial and Quantitative Analysis 12.

Bierwag, G.O., Kaufman, G.G., 1977. Coping with the risk of interest rate fluctuations: A note. Journal of Business 50.

Bierwag, G.O., Kaufman, G.G, 1978. Bond portfolio strategy simulations: A critique. The Journal of Financial and Quantitative Analysis 13.

Bolder, D., Streliski, D., 1999. Yield curve modelling at the Bank of Canada. Technical report no. 84.

Brennan, M.J., Schwartz, E.S., 1983. Duration, bond pricing and portfolio management in Bierwag, G.O., Kaufman, G.G., Toevs, A., eds., Innovations in Bond Portfolio Management: Duration Analysis and Immunization. Contemporary studies in economic and financial analysis 41. JAI Press. Greenwich, Connecticut.

Broadie, M., Sundaresan, S., 1992. The pricing of timing and quality options: an application to Treasury bond and Treasury note futures market. Working Paper. Columbia University.

Cecchetti, S.G., Cumby, R.E., Figlewski, S., 1988. Estimation of the optimal futures hedge. The Review of Economics and Statistics 70, 4.

Chambers, D.R., Carleton, W.T., McEnally, R.W., 1988. Immunizing default-free bond portfolios with a duration vector. Journal of Financial and Quantitative Analysis 23.

Chatfield, C., Collins, A.J., 1980. Introduction to multivariate analysis. Chapman and Hall.

Cox, J., Ingersoll, J., Ross, S., 1981. The relation between forward prices and futures prices. Journal of Financial Economics 9, 4.

Dattatreya, R.E., Fabozzi, F.J., 1995. The risk-point method for measuring and controlling yield curve risk. Financial Analysts Journal, July-August.

Derman, E., 1997. The future of modeling. Risk, December.

Ederington, L.H., 1979. The hedging performance of the new futures market. Journal of Finance 34.

Elton, E., Gruber, M., Rentzler, J., 1983. Intraday tests of the efficiency of the Treasury bill futures market. Review of Economics and Statistics.

Elton, E.J., Gruber, M.J., Nabar P.G. 1988 Bond returns, immunization and the return generating process, Studies in Banking and Finance 5.

Eurex Communications, 2003, Interest rate derivatives - Fixed income trading strategies.

Fabozzi, F., 1998. Treasury Securities and Derivatives. Frank J. Fabozzi Associates, Pennsylvania.

Figlewski, S., 1983. Hedging performance and basis risk in stock index futures. Journal of Finance 39, 3.

Figlewski, S., 1985. Hedging with Financial Futures for Institutional Investors: From Theory to Practice. Cambridge: Ballinger Press.

Fisher, L., Weil, R.L., 1971. Coping with the risk of market-rate fluctuations: Returns to bondholders from naïve and optimal strategies. Journal of Business 44.

Fisher, M., Nychka, D. and Zervos, D., 1995. Fitting the term structure of interest rates with smoothing splines. Finance and Economics Discussion Series, 95-1, Federal Reserve Board.

Fong, H.G., Vasicek, O.A., 1984. A risk minimizing sytategy for portfolio immunization. Journal of Finance 39.

Franckle, C., The Hedging performance of the new futures markets: Comment. Journal of Finance, December.

Fung, W., Hsieh, D.A., 1996. Event risk and risk management of global yield curve exposure. Journal of Fixed Income 6.

Golub, B., Tilman, L., 1997. Measuring yield curve risk using principal components analysis, Value-at-Risk, and key rate durations. Journal of Portfolio Management, Summer.

Grinblatt, M. and Titman, S. (2001) Financial Markets and Corporate Strategy. McGraw-Hill/Irwin, 2nd Edition.

Gultekin, N.B., Rogalski, R.J., 1984. Alternative duration specifications and the measurement of basis risk: empirical tests. Journal of Business 57, 2.

Hilliard, J.E., 1984. Hedging interest rate risk with futures portfolios under term structure effects. Journal of Finance 39, 5.

Hilliard, J.E., 1988. A unified approach to hedging interest rate risk with financial futures. Decision Sciences 19.

Hicks, J.R., 1939. Value and capital. Oxford: Claredon Press.

Ho, T., 1992. Key rate duration: measures of interest rate risks. Journal of Fixed Income 2, 2.

Hull, J.C., 2002, Fundamentals of futures and options markets, 4th edition. Prentice Hall, New Jersey.

Ilmanen, A., 1992. How well does duration measure interest rate risk. The Journal of Fixed Income, March.

Ilmanen, A., 1996. A Framework for analyzing yield curve trades. Salomon Brothers.

Ingersoll J.E., 1983. Is Immunization Feasible? Evidence from the CRSP Data, Innovations in Bond Portfolio Management: Immunization and Duration Analysis, JAI Press.

Johnson, L.L., 1960, The theory of hedging and speculation in commodity futures. Review of Economic Studies 27, 3.

Jones, F.,1991. Yield curve strategies. Journal of Fixed Income, September.

Koenigsberg, M., Bourtzos, Y., 1991. Deciding between futures and cash: A closer look at the basis. Salomon Brothers.

Koutmos, G., Pericli, A., 2000. Are multiple hedging instruments better than one? Journal of Portfolio Management, Winter.

Leschhorn, H., 2001. Managing yield curve risk with combination hedges. Financial Analysts Journal 5-6.

Litterman, R., Scheinkman, J., 1991. Common factors affecting bond returns. Journal of Fixed Income 1, 1.

Macauley, F.R., 1938. Some theoretical problems suggested by the movements of interest rates, bond yields, and stock prices in the U.S. since 1856. National Bureau of Economic Research, New York.

McCulloch, J.H., 1975. The tax-adjusted yield curve. Journal of Finance 30.

Nelson, C.R., Siegel, A.F., 1987. Parsimonious modelling of yield curves. Journal of Business 60.

Plona, C., 1997. The European Bond Basis. The McGraw-Hill Companies, Inc.

Rendleman, R.J., 1999. Duration-based hedging with treasury bond futures. Journal of Portfolio Management, June.

Reitano, R., 1990. Non-parallel yield curve shifts and durational leverage. Journal of Portfolio Management 16, 4.

Reitano, R.R., 1991. Multivariate duration analysis. Transactions of the Society of Actuaries, 43.

Reitano, R.R., 1992. Non-parallel yield curve shifts and immunization. Journal of Portfolio Management, Spring.

Redington, F.M.,1952. Review of the principle of life office valuations. Journal of the Institute of Actuaries 78.

Richard, S., Sundaresan, S., 1981. A continuous time model of forward prices and futures prices in a multigood economy. Journal of Financial Economics 9, 4.

Sundaresan, S., 1997. Fixed income markets and their derivatives. South-Western College Publishing.

Svensson, L., 1994., Estimating and interpreting forward interest rates: Sweden 1992-94. IMF working paper series, No 114.

Wilmott, P., 2001. Paul Wilmott Introduces Quantitative Finance. John Wiley & Sons, Ltd. Chichester, England.

Appendix A

Hedging performance of different models on single German government bonds

The table describes the average hedging performance of five hedging methods, when 15 single German government bonds were hedged using bond futures contracts. The estimation period was 1/1/1999 - 5/31/2002. The figures are averages over the standard deviations and returns from the 15 single bonds. In each case, portfolio consist of a single bond and the corresponding amount of short futures contracts as suggested by the hedging methods. The appropriate amount of futures contracts is recalculated weekly and corresponding changes are made. The return is holding period return for the portfolio excluding transaction cost of futures trades. Coupons are reinvested in the bonds. Standard deviations and returns are annualized.

*In 1999, regression hedge ratios are calculated for period 04/01/1999-12/31/1999 only.

	DBR 6.5 7/15/2003		DBR 6 9/15/2003		DBR 6.75 7/15/2004		DBR 7.5 11/11/2004		DBR 7.375 1/3/2005	
	Std dev.	Return	Std dev.	Return	Std dev.	Return	Std dev.	Return	Std dev.	Return
Duration	0.87%	4.12%	0.82%	3.94%	1.21%	4.23%	1.40%	4.84%	1.43%	4.57%
Regression*	1.08%	3.91%	1.01%	3.89%	1.38%	4.09%	1.42%	4.29%	1.37%	4.54%
PCA	0.82%	4.18%	0.71%	3.94%	1.09%	4.34%	1.23%	4.44%	1.29%	4.41%
Risk point	0.85%	4.26%	0.74%	4.01%	1.12%	4.36%	1.24%	4.44%	1.30%	4.41%
Combination hedge	1.35%	4.30%	0.92%	4.17%	0.98%	4.28%	1.10%	4.40%	1.08%	4.26%
Unhedged	2.34%	3.69%	2.50%	3.45%	2.95%	3.77%	3.23%	3.83%	3.13%	3.78%

	DBR 6.875 5/12/2005		DBR 6.5 10/14/2005		DBR 6 1/5/2006		DBR 6 2/16/2006		DBR 6.25 4/26/2006	
	Std dev.	Return	Std dev.	Return	Std dev.	Return	Std dev.	Return	Std dev.	Return
Duration	1.46%	4.55%	1.24%	4.33%	1.25%	4.02%	1.37%	4.39%	1.66%	4.63%
Regression*	1.54%	4.58%	1.46%	4.35%	1.47%	3.97%	1.52%	3.99%	1.66%	4.18%
PCA	1.33%	4.47%	1.13%	4.29%	0.98%	3.97%	1.08%	4.29%	1.30%	4.26%
Risk point	1.36%	4.45%	1.20%	4.24%	1.04%	3.93%	1.16%	4.26%	1.40%	4.23%
Combination hedge	1.48%	4.48%	1.20%	3.59%	1.35%	4.19%	1.03%	4.50%	1.35%	3.84%
Unhedged	3.41%	3.82%	3.58%	3.58%	3.74%	3.26%	3.83%	3.57%	3.84%	3.50%

	DBR 6 1/4/2007		DBR 6 7/4/2007		DBR 5.25 1/4/2008		DBR 4.125 7/4/2008		DBR 4.75 7/4/2008	
	Std dev.	Return	Std dev.	Return	Std dev.	Return	Std dev.	Return	Std dev.	Return
Duration	1.35%	4.34%	1.45%	3.97%	1.20%	3.88%	0.98%	3.60%	1.08%	3.87%
Regression*	1.60%	4.24%	1.57%	4.62%	1.56%	3.95%	1.54%	3.64%	1.56%	3.81%
PCA	1.11%	4.15%	1.31%	4.38%	0.96%	3.83%	0.79%	3.40%	0.89%	3.66%
Risk point	1.25%	4.18%	1.49%	4.43%	1.17%	3.89%	0.97%	3.49%	1.14%	3.76%
Combination hedge	1.08%	4.19%	1.23%	4.32%	1.07%	3.90%	1.08%	3.51%	1.09%	3.70%
Unhedged	4.17%	3.30%	4.15%	3.47%	4.63%	2.87%	4.82%	2.44%	4.75%	2.67%

Appendix B: Conversion factor calculation for fixed income futures at Eurex

$$CF = \frac{1}{1.06^{f}} \cdot \left[\frac{c}{100} \cdot \frac{\delta_i}{act_2} + \frac{c}{6} \cdot \left(1.06 - \frac{1}{1.06^{n}} \right) + \frac{1}{1.06^{n}} \right] - \frac{c}{100} \cdot \left(\frac{\delta_i}{act_2} - \frac{\delta_e}{act_1} \right)$$

where

CF	conversion factor
δ_e	NCD1y – DD
act_1	NCD – NCD1y, where $\delta_e < 0$
	NCD1y – NCD2y, where $\delta_e \geq 0$
δ_i	NCD1y – LCD
act_2	NCD – NCD1y, where $\delta_i < 0$
	NCD1y – NCD2y, where $\delta_i \geq 0$
f	$1 + \delta_e / act_1$
c	Coupon
n	Integer years from the NCD until the maturity date of the bond
DD	Delivery date
NCD	Next coupon date
NCD1y	1 year before the NCD
NCD2y	2 years before the NCD
LCD	Last coupon date before the delivery date